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Professor R. A. Sampson, M.A., D.Sc., F.R.S., PRESIDENT, in the Chair.

Edward Lindsay Ince, M.A., B.Sc., Trinity College, Cambridge, and

John Cromwell Prior, Rosedene, South Hayling, Hants,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

Mrs. Louisa Petitot Freeman, 67 High Street, Oxford (proposed by A. A. Rambaut);

William Henry Pick, B.Sc. (Lond.), F.C.S., Queen Mary's School, Basingstoke, Hampshire (proposed by Alfred W. Porter); and

J. G. E. G. Voûte, Royal Observatory, Cape of Good Hope (proposed by S. S. Hough).

Mr. J. Jackson, Mr. G. J. Newbegin, and Mr. W. Shackleton were appointed Auditors of the Treasurer's Accounts for 1916.

Forty presents were announced as having been received since the last meeting, including, amongst others:—

Vols. 36 to 107 inclusive of the *Astronomische Nachrichten*, presented by Joseph and Walter Baxendell.

A special vote of thanks was passed to the Messrs. Baxendell for their valuable gift.

On Certain Possible Distributions of Meteoric Bodies in the Solar System. By Harold Jeffreys, B.A., M.Sc.

This investigation deals with problems that arise when a swarm of small solid bodies, of size comparable with a meteor, or even smaller, is supposed to move under gravity about a large nucleus. In all cases the united mass of the particles is supposed sufficiently small for their mutual attractions to be neglected in a first approximation, and the object of the work is to find out in what way the systems will degenerate on account of their mutual impacts.

The first part of the paper deals with cases in which the motion is steady, and the swarm is symmetrically distributed about an axis through the central mass. Two examples that are considered are the rings of Saturn and the zodiacal light. It is shown that in the case of Saturn's rings the frequency of collision is very great, and consequently, on account of the loss of relative motion at every collision, the rings must long ago have reached a state in which all the particles are moving in very accurate circles, all in the same plane. The zodiacal light being assumed to be caused by meteors, it is shown that it must be of later origin than the planets; no information is obtained, however, when the particles causing it are assumed to be small compared with the wave-length of light.

The second part treats of the motion of meteors in the neighbourhood of the earth's orbit. The motion is not assumed to be steady, and the frequency of collision is small, the life of the system being comparable with the age of the earth itself. From the ratio of the numbers of meteors picked up by the earth in front and behind as it moves, it is shown to be highly probable that a large majority of these bodies move in direct orbits.

The third part of the paper considers the Planetesimal Hypothesis of Chamberlin and Moulton. The objections to the Nebular Hypothesis are first summarised, and it is then shown that while the Planetesimal Hypothesis avoids most of them, it is nevertheless open to a new one, namely, that collisions between the small bodies would be so frequent, and would occur with so high a velocity, that the particles would be fused and volatilised long before accretion could produce any important effect on bodies as large as the planets. If the planetesimals be supposed to have moved initially in nearly circular orbits, this objection is avoided, but the planets are required to have always been very large. An alternative hypothesis is, however, suggested, which may combine the advantages of the Nebular and Planetesimal Hypotheses. The original nebula is in this case assumed to be gaseous, with a strong condensation to the centre, and perhaps highly heterogeneous in other respects, but moving in such a way that the period of revolution about the centre depends on the distance from the

centre, so that the criterion of the constancy of angular momentum is automatically satisfied.

I. Problems of Steady Motion.

When a large number of particles are moving in a field of force in such a way that collisions are infrequent and the distribution of mass and velocities with respect to space remains constant, the motion is said to be steady. The condition that collisions are rare makes the problem totally different from that considered by Lockyer, Darwin, and du Ligondès. Mathematically the nearest analogue is the motion of stars in clusters, as treated by Professor Eddington * and Mr. J. H. Jeans.

If N denotes the number of particles within the element of volume $dx\,dy\,dz$ whose velocity components lie between u and $u+du$, v and $v+dv$, w and $w+dw$, we have

$$\mathrm{N} = f\,du\,dv\,dw\,dx\,dy\,dz \quad . \quad . \quad . \quad (1)$$

where f is some function of (u, v, w, x, y, z).

Then it has been shown by Jeans that the condition for steady motion is that $f=$constant is a first integral, independent of the time, of the equations of motion of a particle.† If, however, this is so, f must be a function of the existing first integrals, and conversely any function of these will satisfy the conditions for steady motion.

Now in the case considered the mutual attraction of the particles is neglected in comparison with the attraction of the central body, and consequently the potential is μ/r. There are then four first integrals of the equations of motion, namely, the energy equation and the three momental equations.

Taking spherical polar co-ordinates, we have

$$x = r\sin\theta\cos\phi\,;\quad y = r\sin\theta\sin\phi\,;\quad z = r\cos\theta.$$

The angular momenta are found to be given by

$$h_1 \equiv y\dot{z} - z\dot{y} \equiv -r^2\dot{\theta}\sin\phi - r^2\dot{\phi}\sin\theta\cos\theta\cos\phi \quad . \quad (2)$$

$$h_2 \equiv z\dot{x} - x\dot{z} \equiv r^2\dot{\theta}\cos\phi - r^2\dot{\phi}\sin\theta\cos\theta\sin\phi\,. \quad . \quad (3)$$

$$h_3 \equiv x\dot{y} - \dot{x}y \equiv r^2\sin^2\theta\dot{\phi} \quad . \quad . \quad . \quad . \quad . \quad (4)$$

All these must be constant in the motion of any particle.

Also the equation of energy gives

$$\mathrm{E} \equiv \frac{2\mu}{r} - \dot{x}^2 - \dot{y}^2 - \dot{z}^2 \equiv \frac{2\mu}{r} - \dot{r}^2 - r^2\dot{\theta}^2 - r^2\sin^2\theta\dot{\phi}^2 \quad . \quad (5)$$

Then the most general steady motion is given by making

$$f = \chi(\mathrm{E}, h_1, h_2, h_3) \quad . \quad . \quad . \quad (6)$$

where χ is an arbitrary function.

* *Monthly Notices*, vol. lxxvi., 1915, p. 37.
† Jeans, *Monthly Notices*, vol. lxxvi., 1915, p. 78.

Now we intend to consider only distributions that are symmetrical about an axis, so that f must not involve ϕ explicitly. The functions h_1 and h_2 can therefore only be involved through the sum of their squares, which is independent of ϕ. Then

$$\left.\begin{aligned} f &= \chi(\mathrm{E},\ h_1^2+h_2^2,\ h_3) \\ &= \chi\left(\frac{2\mu}{r}-\dot r^2-r^2\dot\theta^2-r^2\sin^2\theta\dot\phi^2,\ r^4\dot\theta^2+r^4\sin^2\theta\cos^2\theta\dot\phi^2,\ r^2\sin^2\theta\dot\phi\right)\end{aligned}\right\} \quad (7)$$

It follows that in any symmetrical distribution the mean values of $\dot r$ and $\dot\theta$ among the particles at every point must be zero.

(*a*) A very natural type of distribution to try would be one in which f is of the form

$$\chi\{\mathrm{E}+2\alpha h_3-\beta h_3^2-\gamma(h_1^2+h_2^2)\}=\chi(\nu),\ \text{say},$$

where α, β, and γ are constants. Then

$$\nu = 2\mathrm{K}-\dot r^2-r^2\dot\theta^2-r^2\sin^2\theta\dot\phi^2+2\alpha r^2\sin^2\theta\dot\phi - \beta r^4\sin^4\theta\dot\phi^2-\gamma r^4\dot\theta^2-\gamma r^4\sin^2\theta\cos^2\theta\dot\phi^2 \quad (8)$$

where K at present denotes the gravitational potential.

Put

$$\dot r = u,\ r\dot\theta = v,\ r\sin\theta\dot\phi = \frac{\alpha r\sin\theta}{1+\beta r^2\sin^2\theta+\gamma r^2\cos^2\theta}+w \quad . \quad (9)$$

It is seen at once that

$$\nu = \frac{2\mu}{r}-u^2-(1+\gamma r^2)v^2-(1+\beta r^2\sin^2\theta+\gamma r^2\cos^2\theta)w^2 + \frac{\alpha^2r^2\sin^2\theta}{1+\beta r^2\sin^2\theta+\gamma r^2\cos^2\theta} \quad (10)$$

Thus the velocity distribution at any point has ellipsoidal symmetry about the mean velocity that makes $u=v=w=0$. This hypothesis is, however, not satisfactory, for a reason that will be given.

The numerical density at any point is $\iiint f\,du\,dv\,dw$, all the integrations being from $-\infty$ to $+\infty$.

Let

$$2\mathrm{K}+\frac{\alpha^2r^2\sin^2\theta}{1+\beta r^2\sin^2\theta+\gamma r^2\cos^2\theta}=\mathrm{V}^2 \quad . \quad . \quad (11)$$

Now substitute

$$u=\mathrm{V}\xi\,;\ v=\mathrm{V}\eta/(1+\gamma r^2)^{\frac12}\,;\ w=\mathrm{V}\zeta/(1+\beta r^2\sin^2\theta+\gamma r^2\cos^2\theta)^{\frac12} \quad (12)$$

Then the density becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\chi\{\mathrm{V}^2(1-\xi^2-\eta^2-\zeta^2)\}\mathrm{V}^3d\xi\,d\eta\,d\zeta \div(1+\gamma r^2)^{\frac12}(1+\beta r^2\sin^2\theta+\gamma r^2\cos^2\theta)^{\frac12} \quad (13)$$

provided β and γ are not negative.

Now as long as the form of χ is such that the triple integral has any meaning, it is evidently a function of V only.

So far the solution is exact. When r is great, provided the mass is finite, α is not zero, and β and γ are not both zero, V^2 tends to $\alpha^2/(\beta+\gamma\cot^2\theta)$, and is therefore constant for any particular colatitude. The density at a great distance is then of the form $F(\beta+\gamma\cot^2\theta)/r^2\sin\theta$. The total mass between two spheres of radii r and $r+dr$ is then $dr\iint F d\theta d\phi$, and is therefore equal to Mdr where M is a constant: M cannot be zero, as it is physically obvious that F is essentially positive. Thus the condition that the total mass of the system is finite cannot be satisfied.

If for $\chi(\nu)$ we write $e^{h^2\nu}$, where h is a constant, the distribution reduces to the Schwarzschild rotational distribution found by Professor Eddington in the paper already mentioned.

If β and γ are both zero and α is not zero, the mean velocity at any point is $\alpha r\sin\theta$. If then this is greater than $(2\mu/r)^{\frac{1}{2}}$, the particles moving with velocities greater than the mean will leave the system. Hence the system is confined to a region surrounding the axis and bounded by the surface $r^3\sin^2\theta=2\mu/\alpha^2$.

If either β or γ is negative, a region will exist such that at the margin of it infinite velocities are possible without making ν different from V^2.

The argument does not apply if α is 0, corresponding to a state without any general rotation. In that case $V^2=2K$, and forms of χ can be found so that the total mass is finite. This has been done in the case of globular star clusters by Professor Eddington * and Mr. Jeans,† who find that with a law of the type $\nu=E$ it is possible to make the density proportional to r^{-5} at a great distance.

(*b*) Consider next a distribution in which χ is always zero unless Eh_3^2 lies between μ^2 and $\mu^2(1-\alpha)$, where α is a small positive quantity that may be a function of E, $h_1^2+h_2^2$, and h_3.

Put $w=r\sin\theta\dot{\phi}$, so that $h_3=r\sin\theta w$ . . . . (14)

Also put $w=\operatorname{cosec}\theta\sqrt{(\mu/r)}+w'$ . . . . . (15)

Then

$$\frac{\mu^2}{h_3^2}-E=\dot{r}^2+r^2\dot{\theta}^2+\frac{h_3^2}{r^2}\operatorname{cosec}^2\theta-\frac{2\mu}{r}+\frac{\mu^2}{h_3^2}$$

$$=\dot{r}^2+r^2\dot{\theta}^2+\left(\frac{\mu}{h_3}-\frac{h_3}{r}\right)^2+\frac{h_3^2}{r^2}\cot^2\theta \quad . \quad . \quad (16)$$

But $\frac{\mu^2}{h_3^2}-E$ lies between 0 and $\frac{\mu^2\alpha}{h_3^2}$. Hence each of the four squares of which it is the sum must separately be small. If for simplicity we regard α as a small quantity of the *second* order, then it follows that $\dot{r}$, $\dot{\theta}$, $h_3^2-\mu r$, and $\cot\theta$ are all small quantities of the first order; thus all the particles must be moving in very nearly circular orbits close to the invariable plane. This condition alone

* *Monthly Notices*, vol. lxxvi., 1916, p. 579. † *Ibid.*, p. 570.

is therefore enough to define a system in which the orbits are of this type, and may thus give a fair representation of the movement of Saturn's rings and the zodiacal light, if the composition of this is meteoric. Such systems as these are meant when the term "a flat system" is used later in this paper.

If, for instance, we make χ equal to $\mathrm{H}\left(\alpha - 1 + \frac{\mathrm{E}h_3^2}{\mu^2}\right)^k$, when the quantity inside the bracket is positive and less than $\mu^2\alpha$, and zero in all other circumstances, the distribution is of the type considered. H and α may be functions of E and h_3.

Put $\dot{r} = u$, $r\dot{\theta} = v$, and substitute for h_3 in terms of w'.

Then

$$\frac{\mu^2}{h_3^2} - \mathrm{E} = u^2 + v^2 + \left(\frac{2\sqrt{(\mu/r)}\sin\theta w' + \sin^2\theta w'^2}{\sqrt{(\mu/r)} + \sin\theta w'}\right)^2 + \frac{\mu\cot^2\theta}{r}\left(1 + \sqrt{\frac{r}{\mu}}\sin\theta w'\right)^2.$$

When we neglect small quantities of order above the second, this reduces to

$$u^2 + v^2 + 4w'^2 + (\mu/r)\cot^2\theta \quad . \quad . \quad . \quad (17)$$

Hence

$$\chi = \mathrm{H}\left\{\alpha - \frac{r}{\mu}(u^2 + v^2 + 4w'^2) - \cot^2\theta\right\}^k \quad . \quad . \quad (18)$$

Now H and α are functions of E and h_3. For any particular value of r we have therefore

$$\mathrm{H} = \mathrm{H}_0 + w'\left(\frac{\partial \mathrm{H}}{\partial w'}\right)_0 + \tfrac{1}{2}w'^2\left(\frac{\partial^2 \mathrm{H}}{\partial w'^2}\right)_0 + u^2\left(\frac{\partial \mathrm{H}}{\partial u^2}\right)_0 + v^2\left(\frac{\partial \mathrm{H}}{\partial v^2}\right)_0 + \cot^2\theta\left(\frac{\partial \mathrm{H}}{\partial \cot^2\theta}\right)_0,$$

where the suffixes indicate that the quantities are evaluated for those particles for which u, v, w' and $\cot\theta$ are zero. All terms after the first are evidently of at least the first order of small quantities, and hence in (18) H_0 can be written for H. Similarly

$$\alpha = \alpha_0 + w'\left(\frac{\partial \alpha}{\partial w'}\right)_0 + \text{ similar terms}$$

$$= \alpha_0 + r\sin\theta w'\left(\frac{\partial \alpha}{\partial h_3}\right)_0 + \text{ etc.},$$

so that every term after the first is of at least the third order, provided α does not vary too rapidly with h_3.

Hence H and α for any particle can be given their values for a particle at the same distance from the centre, moving in a circular orbit in the equatorial plane. In other words, H and α may be regarded as functions of r only, and not of u, v, w', and θ.

To the first order the mean velocity at any point is equal to the circular velocity, and consequently a swarm of this type cannot affect the mean distance of a planet to any important extent.

The density is

$$\iiint \chi \, du \, dv \, dw = 2^{2k+2}\pi \frac{k!\,(k+1)!}{(2k+3)!}\left(\frac{\mu}{r}\right)^{\frac{3}{2}} \frac{\mathrm{H}(\alpha - \cot^2\theta)^{\frac{2k+3}{2}}}{\sin\theta} \quad (19)$$

provided $\cot^2\theta$ is less than α.

If $\cot^2\theta$ is greater than α, the density is zero.

If α and H were constants, the total mass would evidently be infinite. If, however, H is constant and α is a function of h_3 so that it tends to zero at infinity more rapidly than $h_3^{-(3+2\epsilon)/(k+2)}$ where ϵ is a small positive quantity, the mass is finite. There will therefore be no difficulty in ensuring finiteness of mass.

If $\mathrm{H}\alpha^{k+2}$ is proportional to h_3 between definite limits for h_3, and otherwise zero, the number of particles in a small cross-section of the system parallel to the axis of symmetry is constant between two definite radii, and zero everywhere else. Such a system must have a strong resemblance to Saturn's rings.

The Effect of Collisions in a Flat System.

On the hypothesis that collisions are rare, it is possible to find steady motions that a given type of system is capable of describing. These collisions must, however, occur occasionally, so that the steady state can only be regarded as a first approximation. In investigating the permanence of a system it is therefore necessary to estimate the frequency of collisions and the character of their effect on the system.

Let ν be the number of particles per unit volume at any point, and let the number of particles per unit volume whose component velocities lie respectively between u and $u+du$, v and $v+dv$, w and $w+dw$, be $\nu f(u, v, w)du\,dv\,dw$. Further, if all the particles in any system are supposed to be equal spheres of diameter σ, the probability that a particle with velocities (u_1, v_1, w_1) will have a collision in time dt is

$$dt\iiint\iint \nu\sigma^2 f(u, v, w)\{l(u_1-u)+m(v_1-v)+n(w_1-w)\}du\,dv\,dw\,d\omega$$

where the range of integration for u, v, and w is through all values from $-\infty$ to $+\infty$, subject to the condition that $l(u_1-u)+m(v_1-v)+n(w_1-w)$ is positive. The last two integrations are with regard to a surface element $d\omega$ whose co-ordinates are (l, m, n) and extend over a unit sphere.

In an average element of volume the velocity distribution is given by $\nu f = (r/\mu)^k \mathrm{H}\{(\mu/r)(\alpha - \cot^2\theta) - (u^2+v^2+4w^2)\}^k$ when the quantity in the bracket is positive and is otherwise zero; in this w is now written for w'.

If now we put

$$(\mu/r)(\alpha - \cot^2\theta) = \delta, \quad . \quad . \quad . \quad (20)$$

the first three integrations give, by the theorem proved in the appendix,

$$(r/\mu)^k \frac{\mathrm{H}\sigma^2\pi\delta^{k+2}}{2(k+1)}(l^2+m^2+\tfrac{1}{4}n^2)^{\frac{1}{2}}\left[\frac{(1-\mathrm{L}_0{}^2)^{k+2}}{2(k+2)} - \mathrm{L}_0\int_0^{\theta_0}\sin^{2k+3}\theta\,d\theta\right] \quad (21)$$

where

$$L_0 = (lu_1 + mv_1 + nw_1)/\{\delta(l^2 + m^2 + \tfrac{1}{4}n^2)\}^{\frac{1}{2}}$$

and $\cos\theta_0 = L_0$, subject to the condition that θ_0 lies between 0 and π.

Calling the quantity inside the square brackets $\lambda(L_0)$, and the factor outside them A, then the probability of collision within the elementary solid angle $d\omega$, whose direction cosines are (l, m, n), is $A\lambda(L_0)d\omega\,dt$. Again, the probability of a collision within an equal angle, the direction cosines being $(-l, -m, -n)$, is $A\lambda(-L_0)d\omega\,dt$. The two together give $A\,d\omega\,dt\{\lambda(L_0) + \lambda(-L_0)\}$.

Now

$$\frac{d}{dL_0}\Big\{\lambda(L_0) + \lambda(-L_0)\Big\} = \frac{d}{dL_0}\Big\{\frac{(1 - L_0^2)^{k+2}}{k+2} + L_0\int_{\theta_0}^{\pi-\theta_0}\sin^{2k+3}\theta\,d\theta\Big\}$$

$$= -2L_0(1 - L_0^2)^{k+1} + \int_{\theta_0}^{\pi-\theta_0}\sin^{2k+3}\theta\,d\theta$$

$$+ L_0\sin^{2k+3}(\pi - \theta_0)\frac{d(\pi - \theta_0)}{dL_0} - L_0\sin^{2k+3}\theta_0\frac{d\theta_0}{dL_0}.$$

But

$$\frac{d\theta_0}{dL_0} = -\frac{1}{\sin\theta_0}, \quad \text{and} \quad \sin^{2k+2}\theta_0 = (1 - L_0^2)^{k+1}.$$

Hence the last two terms together give $+2L_0(1 - L_0^2)^{k+1}$, and therefore

$$\frac{d}{dL_0}\{\lambda(L_0) + \lambda(-L_0)\} = \int_{\theta_0}^{\pi-\theta_0}\sin^{2k+3}\theta\,d\theta \quad . \quad . \quad (22)$$

which has the same sign as L_0 and vanishes when $L_0 = 0$. Hence $L_0 = 0$ is the only stationary value of the frequency of collision within a complete cone, and it gives a minimum, namely, $\frac{1}{k+2}$. Thus particles moving with the mean velocity at a point undergo fewer collisions than any others. We shall therefore underestimate the number of collisions if we put $L_0 = 0$. This gives for the probability of collision for a particular particle

$$dt\int\int\frac{(r\mu)^k H\sigma^2\pi\delta^{k+2}(l^2 + m^2 + \frac{1}{4}n^2)^{\frac{1}{2}}}{4(k+1)(k+2)}d\omega \quad . \quad . \quad (23)$$

Also

$$\iint(l^2 + m^2 + \tfrac{1}{4}n^2)^{\frac{1}{2}}d\omega = 4\pi E \quad . \quad . \quad . \quad (24)$$

where E denotes the second complete elliptic integral with modulus $\frac{1}{2}\sqrt{3}$. Its value is given by Greenhill to be 1·212. Hence the probability of collision is greater than

$$1{\cdot}212\,dt(r\mu)^k\pi^2 H\sigma^2\delta^{k+2}/(k+1)(k+2) \quad . \quad . \quad (25)$$

On the other hand, it is easy to see that when $L_0 = 1$, which is the greatest value it can have, $\lambda(L_0) + \lambda(-L_0)$ is $2\int_0^{\pi/2}\sin^{2k+3}\theta\,d\theta$,

which is not overwhelmingly greater than $1/(k+2)$, so that this estimate of the probability of collision will be of the correct order of magnitude for particles for which L_0 is not zero.

Next, it is evident that in a given short interval a particle is more likely to collide with a particle moving in the opposite direction than with one moving in the same direction. The average effect of collisions will therefore be greater than if each particle, instead of colliding with another moving particle, actually collided with a particle moving with the average velocity at the point. The relative velocity of collision is supposed not great enough to cause fusion.

If the coefficient of restitution is λ, the reduction in the velocity relative to the mean in a collision is on the average greater than is expressed by the factor $\frac{1}{2}(1-\lambda)$.

Thus if the average absolute velocity relative to the mean is U initially, then after a time dt it will be reduced to less than

$$U\{1 - \cdot 606(1-\lambda)dt(r\mu)^k\pi^2 H\sigma^2\delta^{k+2}/(k+1)(k+2)\}.$$

Thus

$$\frac{dU}{Udt} = -\frac{\cdot 606\pi^2(1-\lambda)(r\mu)^k H\sigma^2\delta^{k+2}}{(k+1)(k+2)} \quad . \quad . \quad (26)$$

gives a minimum estimate of the rate of degeneration. The actual rate is not likely to be more than five times this in any case.

Application to Saturn's Rings.

No data are available with regard to the actual size of the small bodies believed to compose Saturn's rings, but the product $\nu\sigma^2$ can be estimated from considerations of the luminosity and opacity of these rings. Observations of the satellite Iapetus * have shown that it is quite invisible when it passes into the shadow of either of the main rings, but visible behind Cassini's division and the crape ring. This suggests that the number of particles is such that very little light can get through the rings, owing to reflection, absorption, and scattering. This is confirmed by the high albedo of the rings, which is comparable with that of the body of the planet. The outer ring, however, is not so bright as the inner one.

If we consider a pencil of light from the sun striking the plane of the ring at an angle i, and if the cross-section of the pencil be dS and the thickness T, the number of particles in the part of the ring within this pencil is $\operatorname{cosec} i d\mathrm{S}\int \nu dz$, the integral being taken right through the ring. Each particle stops the light over an area equal to that of its own diametral plane, and if nearly all the light is stopped it follows that $\frac{\pi\sigma^2}{4}\operatorname{cosec} i\int \nu dz$ must be comparable with unity. On the other hand, if this quantity is much greater than unity, light will be stopped and reflected very little more than if it

* Barnard, *Monthly Notices*, vol. l. pp. 107–110.

is only equal to unity, as most of the stoppage will be confined to the first layers. The fact that the inner and outer rings differ in luminosity suggests, then, that the number is not very different from unity for the outer ring, and somewhat greater for the inner one. Young gives* 27° as the value of the inclination of the plane of the rings to the plane of Saturn's orbit, and i must always be equal to or less than this. If ν_0 is the numerical density in the diametral plane,

$$\int \nu dz = 2r\nu_0 \sqrt{a} \int_0^{\pi/2} \cos^{2k+4}\psi \, d\psi.$$

Hence

$$\nu_0 \sigma^2 = 2 \sin i \Big/ \pi r a^{\frac{1}{2}} \int_0^{\pi/2} \cos^{2k+4}\psi \, d\psi$$

for the outer ring. For the inner ring its value must be greater.

Then

$$-\frac{1}{U}\frac{dU}{dt} = \frac{1{\cdot}212(1-\lambda)\sin i}{a^k \int \cos^{2k+4}\psi \, d\psi} \frac{(2k+3)!}{(k+1)!\,(k+2)!} \frac{1}{2^{2k+2}} \left(1 - \frac{\cot^2\theta}{a}\right)^{k+2} \left(\frac{\mu}{r^3}\right)^{\frac{1}{2}} \quad (27)$$

k may have any value, but it is not likely to be negative, on account of the fact that this would prevent the velocities at any point from being concentrated around a mean value. It may, however, be zero. The rest of the numerical coefficient is of order unity. It follows that the time of degeneration at any point is comparable with the time of revolution of the particles at that point, namely, a few hours.

The ring was known to be thin in Galileo's time. Taking its thickness at that date to be less than 10,000 miles, then, since the time of revolution at all points of the ring is less than a day, its present thickness would, from the above estimate, be less than 10,000 exp. (− 100,000) miles, which would be absolutely inappreciable. The hypothesis that the motion is steady would break down before this stage, on account of the frequency of collisions, but it is nevertheless evident that the ring must now be very thin, and practically confined to one plane. The constituent particles must then be distributed in narrow circular strips, each strip being practically a broken solid ring, and perhaps even undergoing something not very different from solid friction from the two on each side of it, which must be moving with slightly different velocities. All estimates of the thickness that have been made visually up to the present are of the nature of upper limits; and from the above it appears very unlikely that any measurable thickness will ever be found. The last visual estimate was made by H. N. Russell,† who states that the thickness must be much less than 13 km.

* *General Astronomy*, p. 392.

† *Astrophysical Journal*, vol. xxvii. p. 230.

The Zodiacal Light.

This consists of a broad luminous band, surrounding the sun, and apparently nearly in the plane of the ecliptic. It extends to more than 90° from the sun, and is quite conspicuous between the orbits of Mercury and Venus. Its cause is not yet quite certain. It is believed to be due to light diffusely reflected or scattered by small particles revolving round the sun, but whether these particles are large or small in comparison with the wave-length of light is not known. Newcomb * states that they are considered to be of meteoric character, but that the final judgment is quite likely to be that the structure is a tenuous gas. If it is gaseous, or composed of particles of dust small compared with the wave-length of light, no progress can be made by the methods here employed, as the amount of light scattered is proportional to $\nu\sigma^6$, which cannot be found without a knowledge of the size of the particles. On the other hand, if it is meteoric, $\nu\sigma^2$ can be at once found from the luminosity. This case will now be considered.

Consider a particle, large compared with the wave-length of light, moving within the earth's orbit and near maximum elongation as seen from the earth. Its distance from the sun is r, and from the earth is ρ. The intensity of solar illumination at the earth being I, and the albedo of the particles p, the amount of light reflected by any particle is $\mathrm{I}(r_0/r)^2\frac{1}{2}\pi\sigma^2 p$, and hence the contribution to the intensity of the light at the earth by the particles in unit volume is $\frac{1}{4}\pi\nu\sigma^2 p\mathrm{I}r_0{}^2/r^2\rho^2$ where r_0 is the earth's mean distance. The further factor $\frac{1}{2}$ has been introduced because only half the visible surface is illuminated. If then we consider a solid angle $d\omega$ as seen from the earth, the intensity within it is $d\omega\int d\rho\,\frac{1}{4}\pi\nu\sigma^2 p\mathrm{I}r_0{}^2/r^2$.

In all, then, the radiation per unit solid angle of the sky is roughly equal to $\frac{1}{4}\pi\nu\sigma^2 p\mathrm{I}r_0{}^2\mathrm{P}/r^2$, where P denotes the depth of the structure in a definite line of observation, and ν and r refer to the point on that line that is nearest to the sun.

If then ω denote the solid angle subtended by a circle of diameter 1°, the above estimate of the intensity gives

$$\frac{\text{total light from an element } \omega \text{ of the sky}}{\text{the solar radiation}} = \frac{\frac{1}{4}\pi\nu\sigma^2 p\mathrm{I}r_0{}^2\mathrm{P}\omega}{\mathrm{I}r^2} \qquad (28)$$

Now Newcomb states † that the luminosity of the "dark" background of the heavens is equivalent to 0·9 stars of magnitude 5·0 per circle of diameter 1°, near the galactic pole, whereas in the Milky Way itself this estimate must be increased by 1·0 to 1·5 for the same area. This excess is to be attributed to the light of the galaxy, and that of the zodiacal light is of about equal intensity. Thus the above fraction is equal to the ratio of the radiation of

* Article "Zodiacal Light," *Ency. Brit.*, 11th edition.
† *Astrophysical Journal*, xiv., 1901, p. 307.

1·5 stars of magnitude 5·0 to the radiation of 1 star of magnitude $-26{\cdot}5$, or $1{\cdot}5 \times 10^{-12{\cdot}6}$.

Then $\omega = \pi^3/129600$; p is taken as $\frac{1}{8}$, equal to the albedo of the moon; $r_0/r = 1{\cdot}4$; $P = 2{\cdot}2 \times 10^{13}$ cm. This estimate of P is obtained by taking the equivalent depth as equal to the diameter of the orbit of Venus.

Then we find

$$\nu\sigma^2 = 3{\cdot}5 \times 10^{-22} \div 1 \text{ cm.} \quad . \quad . \quad . \quad (29)$$

Adopting this value, determining H from it, and substituting in dU/Udt, we find

$$\frac{1}{U}\frac{dU}{dt} = -\frac{{\cdot}606\pi(1-\lambda)(2k+3)!}{(k+1)!\,(k+2)!\,2^{2k+2}}\left(\frac{\mu a}{r}\right)^{\frac{1}{2}}\frac{3{\cdot}5 \times 10^{-22}}{1 \text{ cm.}} \quad . \quad (30)$$

The numerical coefficient is not much different from unity, and a is of order $\frac{1}{4}$ for particles at the distance of Venus.*

The order of the time of degeneration is then found to be 10^7 years, which is short compared with the estimates of the age of the earth derived from radioactivity. The zodiacal light may then be of more recent origin than the planets, and possibly its matter may have been derived from meteors originally moving in more eccentric orbits, or even from the detritus of periodic comets, as has been suggested by Fessenkoff.†

II. The General Distribution of Meteors in the Solar System.

According to Professor Newton,‡ the average distance between the meteors picked up by the earth is about 250 miles. Thus the numerical density ν is $1/6{\cdot}4 \times 10^{22}$ (cm.)^3. Further, the average mass appears to be about 1 grain. Taking the density to be 3, this gives $\sigma = \frac{1}{3}$ cm. roughly. Hence $\nu\sigma^2 = 1{\cdot}7 \times 10^{-24}/1$ cm. This is only $\frac{1}{200}$ of the value of $\nu\sigma^2$ already deduced for the zodiacal light near the orbit of Venus. Thus either the density must increase with great rapidity as we approach the sun, or meteors can account for only a small part of the zodiacal light. Smaller bodies, such as might be produced by the collisions and consequent volatilisation of meteors, would have a larger value of $\nu\sigma^2$ for the same mass.

The rate of degeneration near the earth is of order $\nu\sigma^2 V$, where V is the average velocity with which meteors enter the earth's atmosphere. This is nearly the parabolic velocity of 40 kms./1 sec. Then the time of degeneration is found to be about 5×10^9 years. Nearer the sun it must be much shorter, as both $\nu\sigma^2$ and V are greater. This indicates that in a time that is probably less than the ages of the planets most of the meteors within the orbit of

* Fessenkoff, *Ast. Nach.*, 4693, 1913, vol. cxcvi. p. 229.
† "Sur l'origine de la lumière zodiacale," *Ast. Nach.*, 4752.
‡ Young, *General Astronomy*, art. 771.

Venus would have been volatilised by collision. It appears unlikely, however, that the amount of gas that could thus be produced would scatter enough light to account for the zodiacal light. Thus while the meteoric theory of the zodiacal light is not impossible, it appears to offer no advantage over Fessenkoff's theory that the matter is derived from the detritus of short-period comets, and probably molecular in character.

The Sweeping Action of the Earth.

Consider an element $d\mathrm{S}$ of the surface of the earth, which is supposed spherical. Let U, V, and W be the excesses of the component velocities of the centre of the earth over the mean velocities of the meteors around it, and let the velocity of any individual meteor relative to this mean velocity be u, v, w. Let (l, m, n) be the direction cosines of the outward normal to the surface at the centre of $d\mathrm{S}$.

Then in time dt the element $d\mathrm{S}$ picks up a number

$$d\mathrm{S}\,dt\,vf(u,v,w)du\,dv\,dw\{l(\mathrm{U}-u)+m(\mathrm{V}-v)+n(\mathrm{W}-w)\}$$

of the meteors whose velocities before impact lay within the element $du\,dv\,dw$. This being integrated gives

$$v\int\int\int\int\int\int f(u,v,w)\{l(\mathrm{U}-u)+m(\mathrm{V}-v)+n(\mathrm{W}-w)\}du\,dv\,dw\,d\mathrm{S}\,dt \quad (1)$$

wherein u, v, and w range from $-\infty$ to $+\infty$, subject to the condition that the quantity within the bracket is always positive.

The directions of the axes have not yet been assigned. Suppose them chosen so that $\mathrm{V}=\mathrm{W}=0$. If the distribution of velocities is assumed to be Maxwellian, we have

$$f(u,v,w)=\frac{\alpha^3}{\pi^{\frac{3}{2}}}\exp\{-\alpha^2(u^2+v^2+w^2)\} \quad . \quad . \quad (2)$$

Integrating with regard to u, v, and w, we find that the number of meteors falling per unit time on any area is

$$v\int\int\left\{\frac{\xi_1}{2}(1+\mathrm{Erf}\alpha\xi_1)+\frac{1}{2\alpha\pi^{\frac{1}{2}}}e^{-\alpha^2\xi_1^2}\right\}d\mathrm{S} \quad . \quad . \quad (3)$$

where $\xi_1=l\mathrm{U}$.

The integrand is evidently greater when ξ_1 is positive than when it is negative, so that the number picked up is greater in front than behind.

Let the angular distance from the direction of U be θ. Then $\cos\theta=l$, and $d\mathrm{S}=\mathrm{R}^2\sin\theta\,d\theta\,d\phi=-\mathrm{R}^2dl\,d\phi$, where R is the radius of the earth. The number falling on the area for which l lies between l_1 and l_2 is then

$$\mathrm{R}^2\nu\int_{l_1}^{l_2}\int_0^{2\pi}\left\{\tfrac{1}{2}\mathrm{U}l(1+\mathrm{Erf}\alpha\mathrm{U}l)+\frac{1}{2\alpha\pi^{\frac{1}{2}}}e^{-\alpha^2l^2\mathrm{U}^2}\right\}dl\,d\phi.$$

Putting $\alpha l\mathrm{U} = p$ and integrating with regard to ϕ, this becomes

$$\frac{\pi \mathrm{R}^2 \nu}{\alpha^2 \mathrm{U}^2}\int_{p_1}^{p_2}\left\{p(1+\mathrm{Erf}p)+\frac{1}{\pi^{\frac{1}{2}}}e^{-p^2}\right\}dp$$
$$=\frac{\pi \mathrm{R}^2\nu}{\alpha^2\mathrm{U}^2}\left\{\tfrac{1}{2}p^2+\tfrac{1}{4}\mathrm{Erf}p+\tfrac{1}{2}p^2\mathrm{Erf}p+\frac{1}{2\pi^{\frac{1}{2}}}pe^{-p^2}\right\}_{p_1}^{p_2} \quad (4)$$

(1) To get the ratio between the numbers falling in front and behind, we see that the limits in the two cases are 0 to $\alpha\mathrm{U}$ and $-\alpha\mathrm{U}$ to 0 respectively. Then

Number falling in front hemisphere ÷ number falling in back hemisphere

$$\frac{\alpha^2\mathrm{U}^2+\tfrac{1}{2}\mathrm{Erf}\alpha\mathrm{U}+\alpha^2\mathrm{U}^2\mathrm{Erf}\alpha\mathrm{U}+\pi^{-\frac{1}{2}}\alpha\mathrm{U}\exp(-\alpha^2\mathrm{U}^2)}{-\alpha^2\mathrm{U}^2+\tfrac{1}{2}\mathrm{Erf}\alpha\mathrm{U}+\alpha^2\mathrm{U}^2\mathrm{Erf}\alpha\mathrm{U}+\pi^{-\frac{1}{2}}\alpha\mathrm{U}\exp(-\alpha^2\mathrm{U}^2)} \quad (5)$$

For first magnitude shooting stars the value of this ratio is given by Denning * to be $\frac{64{\cdot}5}{35{\cdot}5}$. This corresponds to the case of $\alpha\mathrm{U} = {\cdot}3394$.

It is evident from the positive sign of this quantity that the mean transverse velocity of the meteors near the earth is less than the earth's velocity. On the other hand, if it were zero, and α were determined by the above condition, the number of meteors with velocities greater than $\mathrm{U}\sqrt{2}$, the parabolic velocity, would be

$$\nu\{1+2^{\frac{3}{2}}\alpha\pi^{-\frac{1}{2}}\mathrm{U}\exp(-2\alpha^2\mathrm{U}^2)-\mathrm{Erf}\alpha\mathrm{U}\sqrt{2}\}={\cdot}926\nu \quad (6)$$

Thus the majority of the meteors would be moving in parabolic or hyperbolic orbits, and therefore could not be true members of the system. The same would hold even if the law of velocity-distribution was considerably altered. It is evident, then, that the mean transverse velocity of the meteors must be in the direction in which the earth moves, and therefore the great majority of the meteors must move in direct orbits.

(2) As the velocity ellipsoid is symmetrical about the axis of U, we can see that the amounts of momentum in the directions of V and W communicated to the earth are zero. The amount carried by the particles picked up by the element $d\mathrm{S}$ in time dt is evidently

$$\nu\, d\mathrm{S}\, dt\iiint m\alpha^3\pi^{-\frac{3}{2}}\exp\{-\alpha^2(u^2+v^2+w^2)\}$$
$$\{l(\mathrm{U}-u)+m(\mathrm{V}-v)+n(\mathrm{W}-w)\}u\,du\,dv\,dw \quad (7)$$

Turn the axes so that the axis of ξ is parallel to the direction (l, m, n). Then

$$\left.\begin{aligned}\xi&=lu+mv+nw\\ \eta&=l'u+m'v+n'w\\ \zeta&=l''u+m''v+n''w\end{aligned}\right\}\text{with the relation}\begin{vmatrix} l & m & n\\ l' & m' & n'\\ l'' & m'' & n''\end{vmatrix}=1.$$

* *Monthly Notices*, vol. liv. p. 540.

Then the integral transforms into

$$\int_{-\infty}^{\xi_1}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m\alpha^3\pi^{-\frac{3}{2}}\exp\{-\alpha^2(\xi^2+\eta^2+\zeta^2)\}(\xi_1-\xi)(l\xi+l'\eta+l''\zeta)d\xi\,d\eta\,d\zeta \quad (8)$$

Now

$$\int_{-\infty}^{\infty} e^{-\alpha^2\eta^2}\eta\,d\eta = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} e^{-\alpha^2\zeta^2}\zeta\,d\zeta = 0.$$

Hence the integral becomes

$$ma l\pi^{-\frac{1}{2}}\int_{-\infty}^{\xi_1} e^{-\alpha^2\xi^2}(\xi_1\xi-\xi^2)d\xi = -\frac{ml}{4\alpha^2}\left[1+\mathrm{Erf}\alpha\xi_1\right].$$

The total momentum acquired by the sphere in time dt is then

$$-\frac{\nu m}{4\alpha^2}\mathrm{R}^2dt\int_{-1}^{1}\int_{0}^{2\pi} l(1+\mathrm{Erf}\alpha l\mathrm{U})dl\,d\phi \quad . \quad . \quad (9)$$

$$=\frac{\pi\nu m\mathrm{R}^2dt}{2\alpha^4\mathrm{U}^2}\left[(\alpha^2\mathrm{U}^2-\tfrac{1}{2})\mathrm{Erf}\alpha\mathrm{U}+\pi^{-\frac{1}{2}}\alpha\mathrm{U}\exp(-\alpha^2\mathrm{U}^2)\right] \quad . \quad (10)$$

Hence in time dt the mass of the earth changes from

$$\mathrm{M} \text{ to } \mathrm{M}+\frac{\pi\mathrm{R}^2\nu m}{\alpha^2\mathrm{U}}\left\{\tfrac{1}{2}\mathrm{Erf}\alpha\mathrm{U}+\alpha^2\mathrm{U}^2\mathrm{Erf}\alpha\mathrm{U}+\pi^{-\frac{1}{2}}\alpha\mathrm{U}e^{-\alpha^2\mathrm{U}^2}\right\}dt = \mathrm{M}+\delta\mathrm{M}, \text{ say,}$$

and the momentum from

$$\mathrm{MU} \text{ to } \mathrm{MU}-\frac{\pi\mathrm{R}^2\nu m}{\alpha^4\mathrm{U}^2}\left\{(\alpha^2\mathrm{U}^2-\tfrac{1}{2})\mathrm{Erf}\alpha\mathrm{U}+\pi^{-\frac{1}{2}}\alpha\mathrm{U}e^{-\alpha^2\mathrm{U}^2}\right\}dt = \mathrm{MU}+\delta(\mathrm{MU}), \text{ say.}$$

Hence the velocity changes from

$$\mathrm{U} \text{ to } \{\mathrm{MU}+\delta(\mathrm{MU})\}/(\mathrm{M}+\delta\mathrm{M}) = \mathrm{U}+\frac{1}{\mathrm{M}}\delta(\mathrm{MU})-\frac{\delta\mathrm{M}}{\mathrm{M}}\mathrm{U}.$$

The effect of the meteors is to modify the orbit in the same way as a tangential retardation of amount

$$\frac{\pi\mathrm{R}^2\nu m}{4\alpha^4\mathrm{U}^2\mathrm{M}}\left\{(4\alpha^4\mathrm{U}^4+4\alpha^2\mathrm{U}^2-1)\mathrm{Erf}\alpha\mathrm{U}+\frac{2}{\sqrt{\pi}}\alpha\mathrm{U}(1+2\alpha^2\mathrm{U}^2)e^{-\alpha^2\mathrm{U}^2}\right\} \quad (11)$$

It is interesting to note that in a case where U is small the expansion of this in powers of U starts with $\frac{8}{3}\sqrt{\pi}\cdot\frac{\mathrm{R}^2\nu m\mathrm{U}}{a\mathrm{M}}$, so that it has not a singularity at $\mathrm{U}=0$. In this case the portion of the retardation due to increase of mass is only $\frac{1}{4}$ of the whole, the rest being due to diminution of momentum.

Increase of Sweeping Power of a Planet due to Gravity.

Owing to the attraction of the planet on a meteor, the orbit of this in the neighbourhood of a planet becomes curved towards it, and consequently more meteors will strike the planet than would be the case if gravity did not act. In calculating the

effect, it will be legitimate to treat the planet as at rest, and the meteors as initially moving in straight lines relative to it. Before a meteor comes within the influence of the planet, let its velocity be V, and the perpendicular from the centre of the planet to its path be p. Then, referred to the centre of the planet as origin, the motion of the particle is given by

$$\dot{r}^2 + r^2\dot{\theta}^2 = \frac{2\mu}{r} + \mathrm{V}^2 \quad . \quad . \quad . \quad . \quad (12)$$

$$r^2\dot{\theta} = p\mathrm{V} \quad . \quad . \quad . \quad . \quad . \quad (13)$$

Within the sphere of influence of the planet the solar attraction is supposed the same for both bodies, and hence does not appear in the equations of motion. If the least distance between the centres is λ, the condition that $\dot{r} = 0$ when $r = \lambda$ gives

$$p^2 = \frac{2\mu\lambda}{\mathrm{V}^2} + \lambda^2 \quad . \quad . \quad . \quad . \quad (14)$$

The condition for a collision is that $\lambda < \mathrm{R}$, the radius of the planet. Hence if $p^2 < \frac{2\mu\mathrm{R}}{\mathrm{V}^2} + \mathrm{R}^2$, there will be a collision. The maximum value of p for a collision to be possible is thus determined as a function of the initial relative velocity. Let it be denoted by P.

Now suppose the distribution of meteoric velocities to be a Maxwellian one about a mean, the velocity of the planet relative to this mean being U; we have

$$f(u, v, w) = \alpha^3\pi^{-\frac{3}{2}} \exp\{-\alpha^2(u^2 + v^2 + w^2)\},$$

and the mean value of P^2 is given by

$$[\mathrm{P}^2]_{\text{mean}} = \alpha^3\pi^{-\frac{3}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{\mathrm{R}^2 + \frac{2\mu\mathrm{R}}{\mathrm{V}^2}\right\} e^{-\alpha^2(u^2+v^2+w^2)}\, du\, dv\, dw \quad (15)$$

where V^2 is now $(\mathrm{U} - u)^2 + v^2 + w^2$.

Put

$$\left.\begin{aligned} \mathrm{U} - u &= \mathrm{V}\cos\chi \\ v &= \mathrm{V}\sin\chi\sin\psi \\ w &= \mathrm{V}\sin\chi\cos\psi \end{aligned}\right\} \quad . \quad . \quad . \quad (16)$$

and $\cos\chi = l$.

Then this becomes

$$\begin{aligned} \alpha^3\pi^{-\frac{3}{2}} \int_0^{\infty}\int_{-1}^{1}\int_0^{2\pi} \left(\mathrm{R}^2 + \frac{2\mu\mathrm{R}}{\mathrm{V}^2}\right) &\exp\{-\alpha^2(\mathrm{V}^2 - 2\mathrm{U}\mathrm{V}l + \mathrm{U}^2)\}\mathrm{V}^2\, d\mathrm{V}\, dl\, d\psi \\ &= \alpha\pi^{-\frac{1}{2}} \int_0^{\infty} \frac{(\mathrm{R}^2\mathrm{V}^2 + 2\mu\mathrm{R})}{\mathrm{U}\mathrm{V}} [\exp\{-\alpha^2(\mathrm{V} - \mathrm{U})^2\} \\ &\qquad - \exp\{-\alpha^2(\mathrm{V} + \mathrm{U})^2\}]\, d\mathrm{V} \\ &= \frac{\pi^{-\frac{1}{2}}\mathrm{R}}{c} \int_0^{\infty} \left(\mathrm{R}z + \frac{2\mu\alpha^2}{z}\right) \{e^{-(z-c)^2} - e^{-(z+c)^2}\}\, dz \end{aligned}$$

where $\alpha\mathrm{V} = z$, $\alpha\mathrm{U} = c$.

$$\int_0^\infty \mathrm{R}z\{e^{-(z-c)^2} - e^{-(z+c)^2}\}dz = c\mathrm{R}\pi^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad (17)$$

$$\int_0^\infty \frac{dz}{z}\{e^{-(z-c)^2} - e^{-(z+c)^2}\}dz = 2e^{-c^2}\int_0^\infty \frac{dz}{z}\sinh(2cz)e^{-z^2}$$

Expanding $\sinh 2cz$, we get *

$$2e^{-c^2}\left[2c\int_0^\infty e^{-z^2}dz + \frac{(2c)^3}{3!}\int_0^\infty z^2e^{-z^2}dz + \ldots\right]$$

$$= 2\pi^{\frac{1}{2}}e^{-c^2}\left[c + \frac{c^3}{3} + \frac{c^5}{2!5} + \frac{c^7}{3!7} + \ldots\right]$$

$$= 2\pi^{\frac{1}{2}}e^{-c^2}\int_0^c e^{x^2}dx \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

Then

$$[\mathrm{P}^2]_{\text{mean}} = \mathrm{R}^2 + 4\mu a^2\mathrm{R}\cdot\frac{1}{c}e^{-c^2}\int_0^c e^{-x^2}dx \quad . \quad . \quad (19)$$

Putting for c its value ·339, found already, this gives

$$[\mathrm{P}^2]_{\text{mean}} \div \mathrm{R}^2 = 1 + 3{\cdot}62ga^2\mathrm{R} \quad . \quad . \quad . \quad (20)$$

Now $2g\mathrm{R}$ is the square of the velocity of a body at the earth's surface that would make it leave the earth. This is known to be about 11 kms./1 sec. The information has already been obtained that $a\mathrm{U} = {\cdot}339$, while U is a considerable fraction of the earth's orbital velocity, as may be seen from the great radial velocities of meteors. Hence $1/a$ is not likely to be less than 40 kms./1 sec. Thus the number of meteors picked up by the earth is increased by less than a tenth of its amount when gravity is taken into account, so that the result already stated, that the majority of the meteors in the solar system must move in direct orbits, is not affected by the fact that gravity was omitted in calculating the value of $a\mathrm{U}$.

III. The Planetesimal Hypothesis.

The Nebular Hypothesis of Laplace.

This hypothesis supposed the solar system to have formerly consisted of a mass of gas, extending beyond the orbit of Neptune, and rotating like a rigid body. In the process of condensation the mass would contract, its angular velocity would correspondingly increase, and ultimately become too great for the equatorial portion to be retained by gravity. This would then be left behind in the form of a ring, which, it was believed, would subsequently collect into a planet. By the repetition of the process the planets were supposed to have been formed; in a

* The method is the same as is employed by Bromwich, *Infinite Series*, p. 473, ex. 33.

similar way the satellites might have been formed from the planets. This hypothesis has been criticised on several grounds, notably by Professor F. R. Moulton.

(*a*) In the first place, the angular momentum of the system, when the nebula extended to the orbit of a particular planet, should correspond to an angular velocity just sufficient to make the centrifugal force balance gravity at the equatorial boundary, in order that any matter should be detached. Now, as angular momentum is a constant, it is possible to calculate what the speed of rotation must have been when the gaseous mass extended to any planetary orbit, and it is actually found to be very much too small to cause any detachment. This criterion has been stated in various forms, first by Babinet, and later by Moulton * and See.† Poincaré considered ‡ that this objection could be met by supposing that very much the greater part of the mass was condensed to a central point before the annulation commenced. The total mass of the surrounding nebula could only be about $\frac{1}{600}$ of the mass of the sun, so that practically the whole of it must have condensed into planets.

(*b*) As contraction would be a continuous process, it follows that instead of the detached matter forming definite rings, it would produce a continuous system of molecules, moving in elliptical orbits, apart from collisions.

(*c*) A ring of matter of the type described by Laplace could not condense into a planet. If it condensed at all, it would produce several planets of about the same size and with the same mean motion, and these could not unite into one.

(*d*) On Laplace's hypothesis every planet must rotate more rapidly than any of its satellites revolve. This condition is violated by Phobos, the inner satellite of Mars. Sir G. H. Darwin's work on tidal friction explained this anomaly on the hypothesis that the solar tides in Mars have slackened Mars's rotation so much as to make its period longer than the period of revolution of Phobos. A similar objection, however, has been again raised by Moulton, who points out that the same condition is violated by the particles that compose the inner edge of Saturn's ring, where tidal friction can scarcely be of sufficient importance to account for the discrepancy.

(*e*) All satellites ought to revolve in the same direction as their primaries rotate. This condition is not satisfied by the outermost satellite of Saturn, and the two outermost satellites of Jupiter. If planets originally had a retrograde rotation, and this was subsequently reversed by tidal friction, these facts might be accounted for; so likewise might the retrograde revolutions of the satellites of Uranus and Neptune, but it remains doubtful whether tidal friction is of sufficient importance at the distances of Jupiter and Saturn to have reversed their rotations. Mr.

* *Astrophysical Journal*, vol. xi., 1900, p. 126.
† *Ast. Nach.*, vol. clxxx., No. 4308, p. 185, 1909.
‡ *Leçons sur les Hypothèses Cosmogoniques*, p. 18.

J. Jackson * has pointed out that at a great distance from a planet a retrograde satellite orbit would be less perturbed by the sun than a direct one would be. This gives an adequate reason for the persistence of these orbits, when once they had commenced, but how they were produced in the first place remains unexplained.

The Planetesimal Hypothesis.

Lockyer, in his work *The Meteoritic Hypothesis*, suggested that nebulæ were composed of swarms of meteors, moving irregularly, and liable to frequent collisions. A similar hypothesis has been advanced by M. du Ligondès. It seems, however, that such a swarm of meteors would behave very like a gas,† and that its method of evolution would also not be very different. It indeed seems likely that it would actually turn into a gaseous nebula very soon.

A hypothesis advanced more recently by Chamberlin and Moulton,‡ called the "Planetesimal Hypothesis," seems open to fewer objections than the older theories. The original nebula is supposed to be of spiral form, and composed of solid or liquid particles called planetesimals moving freely in elliptic orbits about a large nucleus, the two arms of the spiral coinciding roughly with the locus of the aphelia. Several large "knots," or parts of excessive density, were supposed to lie on the arms. In their revolution these are considered to have swept up most of the small particles, and in the process to have developed a direct rotation, for the most part, and to have had the eccentricities of their orbits reduced. The first four objections offered against the theory of Laplace do not arise against this hypothesis. The satellites, according to this theory, grew from smaller nuclei close to the large planetary nuclei. It is stated by Moulton on p. 481 of his *Astronomy* (*infra*) that those with retrograde revolutions would be more resisted by the scattered material than would the direct moving ones, that their orbits would consequently shrink more rapidly, and that they would ultimately be for the most part precipitated on their primaries. From this point of view it seems remarkable that all the known retrograde satellites are on the *outside* of their systems. If the eccentricities were small, it seems more likely that in the neighbourhood of a planet most of the planetesimals nearer the sun would be moving faster than the planet, and those further away would be moving more slowly. The medium as a whole, then, in the neighbourhood of a planet would have a retrograde motion relative to that planet. The retrograde satellites would then approach the primaries less rapidly than would the direct ones, and might even recede. Their present positions can thus be explained on the planetesimal hypothesis

* *Monthly Notices*, vol. lxxiv., 1913, p. 69.

† Sir G. H. Darwin, *Sci. Papers*, vol. iv., or *Phil. Trans.*, 1888.

‡ Chamberlin and Salisbury, *Textbook of Geology*, vol. ii.; Moulton, *Introduction to Astronomy*.

more easily than by Moulton's method, which assumes the eccentricities of the planetesimal orbits to have been fairly great throughout the process, and gives exactly the opposite result.

The difficulty in the way of accepting the hypothesis in its present form arises from the statement that the planetesimals were solid or liquid particles. The spiral form of the nebula obviously could not have been retained for more than a few years at the outside, unless the orbits were highly eccentric. Now let us consider the probable course of events when two solid bodies impinge. When the relative velocity along the normal at the point of incidence is of the order of magnitude that we ordinarily produce in the laboratory, the two bodies rebound, with a smaller relative velocity than before impact. A greater velocity, about that of a rifle bullet, may cause partial fusion, and in this case the bodies may unite into one. With still higher velocities, enough energy may be transformed into heat to convert the bodies into vapour.

Let each of the two bodies be of mass m, and let the relative velocity along the normal be $2v$. Then the energy liberated if they unite is mv^2, so that each unit of mass will receive an amount of heat equal to $v^2/2J$, where J is the mechanical equivalent of heat. Now 400 calories per gram would be sufficient to volatilise most solids. Thus the value of v necessary to vaporise may be taken to be 6 kilometres per second, a comparatively small planetary velocity.

Next, let M and (UVW) be the mass and components of velocity of a large body, and m the mass of a smaller body whose component velocities are (u, v, w), the axis of U being along the normal at the point of contact. If they impinge and unite, the velocity afterwards has components $\left(\frac{MU+mu}{M+m}, \frac{MV+mv}{M+m}, \frac{MW+mw}{M+m}\right)$.

It is then easy to verify that if the bodies were moving before impact in anything like the same direction, which is a condition of the planetesimal hypothesis, the resultant velocity after impact would be intermediate between those before impact, and consequently the major axis of the orbit of the combined body would be between those of the components before their union. This seems to be necessarily true in the great majority of cases. Thus if the planet Mercury was formed according to the planetesimal manner, a large number of planetesimals must have formerly existed whose orbits had major axes less than that of Mercury. Similarly many must have existed whose mean distances were greater than that of Neptune. Further, the orbital eccentricities of the planets, and also those of the planetesimals, must have been large, probably greater than 0·5 originally.

Under these conditions the relative velocity of impact between planetesimals near the orbit of Mercury must have been of the same order of magnitude as half that planet's orbital velocity, or 20 kilometres per second, even in the case postulated by Chamberlin and Moulton, that the collisions were all overtakes. This is above

the velocity required for vaporisation, by what has already been shown. Indeed, observations of meteors at the surface of the earth are by themselves adequate to show that impact at such a velocity would volatilise the bodies concerned. The condition that all the collisions are overtakes seems itself to be very uncertain. On account of the attractions of the planetary nuclei, the perihelia and nodes of the planetesimals must have advanced or retrograded, and it is highly improbable that all would move at the same rate. Thus after a time comparable with 100,000 years the original spiral state would be converted into a state where the motions of the particles near any point would be in widely different directions. Now, as the probability of a collision between two particles is greater the greater is their relative velocity, within certain limits, it follows that in most impacts taking place after that interval the relative velocity must have been comparable with the velocity of a particle in a circular orbit at the same point. Thus in nearly all impacts occurring within the orbit of Jupiter the planetesimals must have turned to gas.

A possible objection to the above argument is that collisions between planetesimals must have been very rare, and consequently that the planets would have time to go through their evolution before the swarm was destroyed. This, however, does not appear to be the case. Let a body of class A be one whose component velocities lie between u and $u+du$, v and $v+dv$, w and $w+dw$ respectively, and one of class B be one whose velocities are between u' and $u'+du'$, v' and $v'+dv'$, w' and $w'+dw'$ respectively, and the number of bodies of class A be $\nu f(uvw)du\,dv\,dw$, where ν is the number of bodies per unit volume. Further, let both bodies be spherical, for greater definiteness, and let (l, m, n) be the direction cosines of the normal at the point of contact, supposed to lie within a small solid angle $d\omega$ at the centre of the first. Then the number of collisions occurring in unit time in unit volume, one body belonging to class A and the other to class B, and the normal lying inside $d\omega$, is

$$\nu^2 f(uvw)f(u'v'w')\mathrm{F}\sigma^2\,du\,dv\,dw\,du'\,dv'\,dw'\,d\omega$$

where σ is the distance between the centres of the bodies and

$$\mathrm{F} = l(u-u') + m(v-v') + n(w-w'),$$

and is thus the normal component of the relative velocity.

Now, in the case of a collision between planetesimals, σ is practically the mean of their diameters, but when a planet picks up a planetesimal, σ is the radius of the planet. Then suppose we consider any particle of class A, its expectation of a collision in unit time with another planetesimal is

$$\iiiint \nu f(u'v'w')\mathrm{F}\sigma^2 du'\,dv'\,dw'\,d\omega.$$

As $\iiint f(uvw)du\,dv\,dw = 1$, this is of the same order of magnitude as $4\pi\nu\sigma^2\sqrt{u^2+v^2+w^2}$. Again, the expectation of

collision with a planet in unit time is similarly of the order of $4\pi N a^2 \sqrt{u^2+v^2+w^2}$, velocities being measured relatively to the movement in a circular orbit at the point considered, and $1/N$ denoting the volume of the region that the hypothesis requires the planet to sweep out.

If then each planetesimal has a diameter 10^{-4} of that of the planet which is likely to absorb it, and the increase of the mass of the planet by accretion is to be as little as 10^{-3}, both these estimates being very conservative, we see on averaging through the region just mentioned that $\nu/N = 10^{-3}a^3/\sigma^3$, each side being the number of particles within it, so that the number of collisions between planetesimals is about ten times the number of planetesimals picked up by the planet. If the planetesimals are to be smaller and the amount of growth greater, the importance of collisions between planetesimals must be greater still, so that there seems to be little doubt that if eccentricities were at all high at the commencement the planetesimal system must have turned to gas long before the planets could grow appreciably by accretion.

It may be remarked that as in impact the relative velocity of the bodies is converted into molecular motion, and the mass is the same before and after collision, the average molecular velocity in the gas produced by impact must be less than a planetary velocity, and consequently very few molecules could leave the system. The system would then presumably degenerate to something not unlike a Laplacian nebula with a very strong central condensation and some pronounced secondary nuclei. This nebula would act on the planetary nuclei as a resisting medium would, and the eccentricities of their orbits would be diminished secularly in the well-known manner.

An alternative to the hypothesis of complete vaporisation is that the colliding particles may be broken up into solid fragments, which would then fly off with velocities comparable with the relative velocity before impact. As, however, the fragments would together have a surface greater than the original particles, and likewise similar velocities relative to each other, it follows that in the degenerate nebula collisions would be much more frequent than in the original one; thus the process would be repeated more and more rapidly until the conversion to vapour was complete. This alternative, therefore, only postpones the difficulty without avoiding it.

The Case of Initially Small Eccentricities.

If all the particles were initially moving in nearly circular orbits in one plane, the relative velocity at impact would be small and they would not be volatilised. In such a case as this it would apparently be impossible to regard existing meteors with their high eccentricities as a survival of the planetesimals, and they must be supposed to be of later origin. As before, collisions between the small bodies would cause their orbits to tend rapidly

to the circular form, whether their collisions took the form of imperfectly elastic rebounds or accretions. The spiral structure is in this case impossible; the eccentricities of the planetary orbits, though small, are still supposed to exist, but their squares are neglected. This hypothesis has the advantage over the standard form of the planetesimal hypothesis that retrograde satellite orbits are more easily accounted for; the most evident defect is that it resembles no known form of system, and its inception would be difficult to account for, unless it arose from the degeneration of a Laplacian nebula.

Consider a planet J moving in such a system of planetesimals. Let the major axis of its orbit be a, and its mean motion n. Further, let μ be the product of the gravitational constant and the mass of the sun.

Let S be the sun. From S draw a line SO whose length is a, moving about S with constant angular velocity n, and nearly coinciding initially with SJ. Let its production OX be the axis of X, and take the axis of Y in the plane of the system. If x and y be the co-ordinates of J relative to these moving axes, its real velocities are $\dot{x} - ny$, $na + \dot{y} + nx$.

The equations of motion are then

$$\left.\begin{aligned} \frac{d}{dt}(\dot{x} - ny) - (na + \dot{y} + nx)n &= \frac{\partial}{\partial x}\left(\frac{\mu}{r}\right) + \mathrm{X} \\ \frac{d}{dt}(\dot{y} + nx + na) + (\dot{x} - ny)n &= \frac{\partial}{\partial y}\left(\frac{\mu}{r}\right) + \mathrm{Y} \end{aligned}\right\} \quad . \quad . \quad (1)$$

where $r = \mathrm{SJ}$, and X and Y are the components of the frictional acceleration. They are proportional to the velocities of J relative to the planetesimals in its immediate neighbourhood.

Now if in a short time dt J picks up a mass dm, and if the mean velocity of the particles constituting dm was $(\bar{u}, \bar{v})$ before impact, then the gain of momentum is $(\bar{u}\,dm, \bar{v}\,dm)$. The gain of mass is dm. Let (U, V) be the velocity of J. Then the change of velocity that friction would produce if acting alone has for its x-component the value

$$\frac{m(\mathrm{U}) + \bar{u}dm}{m + dm} - \mathrm{U} = -(\mathrm{U} - \bar{u})\frac{dm}{m} \text{ to the first order.}$$

Hence the accelerations due to accretion are

$$\left\{ -\frac{(\mathrm{U} - \bar{u})}{m}\frac{dm}{dt}, \quad -\frac{(\mathrm{V} - \bar{v})}{m}\frac{dm}{dt} \right\} \quad . \quad . \quad (2)$$

If all the particles picked up were moving in exactly circular orbits, then the velocities of all of them would be the same, namely, the circular velocity. Actually, however, their velocities are distributed about this value as a mean. If the planet is moving more rapidly than the mean of the particles, then it tends to pick up more of those particles moving less rapidly than the mean than of those moving more rapidly than the mean; so that

the accelerations will be underestimated if we put $(\bar{u}, \bar{v})$ equal to the circular velocities. It has been shown that if the velocity distribution were Maxwellian about the mean, and the planet's velocity relative to the mean were small, the actual accelerations would be four times those given on the hypothesis here adopted.

Now the mean velocity of the particles at J is $\sqrt{\mu/r}$ in the direction at right angles to SJ, whose direction cosines are $\left(-\frac{y}{r}, +\frac{a+x}{r}\right)$.

Its x-component is then

$$-\frac{\mu^{\frac{1}{2}}y}{\{(a+x)^2+y^2\}^{\frac{3}{4}}} = -\frac{\mu^{\frac{1}{2}}y}{a^{\frac{3}{2}}} = -ny \text{ to the first order,}$$

and its y-component is

$$\frac{\mu^{\frac{1}{2}}(a+x)}{\{(a+x)^2+y^2\}^{\frac{3}{4}}} = na - \tfrac{1}{2}nx \quad . \quad . \quad . \quad (3)$$

We can then write for the accelerations X and Y the values

$$-k\dot{x}, \ -k(\dot{y}+\tfrac{3}{2}nx), \text{ where } k \text{ is greater than } \frac{1}{m}\frac{dm}{dt} \quad . \quad (4)$$

Again,

$$\frac{1}{r} = \frac{1}{\{(a+x)^2+y^2\}^{\frac{1}{2}}} = \frac{1}{a}\left(1 - \frac{x}{a} + \frac{x^2}{a^2} - \frac{1}{2}\frac{y^2}{a^2}\right) \text{ to the second order,}$$

and thus the equations of motion become

$$\left.\begin{aligned} \ddot{x} - 2n\dot{y} - n^2a - n^2x &= -\frac{\mu}{a^3}(a-2x) - k\dot{x} \\ \ddot{y} + 2n\dot{x} - n^2y &= -\frac{\mu y}{a^3} - k(\dot{y}+\tfrac{3}{2}nx) \end{aligned}\right\} \quad . \quad . \quad (5)$$

These reduce to

$$\left.\begin{aligned} \ddot{x} - 2n\dot{y} - 3n^2x + k\dot{x} &= 0 \\ \ddot{y} + 2n\dot{x} + k(\dot{y}+\tfrac{3}{2}nx) &= 0 \end{aligned}\right\} \quad . \quad . \quad . \quad (6)$$

These are the equations of motion of a dissipative system in which

$$\left.\begin{aligned} 2\mathrm{R} &= \dot{x}^2 + \dot{y}^2 + 2nx\dot{y} - 2ny\dot{x} + 3n^2x^2 \\ 2\mathrm{F} &= k\dot{x}^2 + k\dot{y}^2 + 3nkx\dot{y} \end{aligned}\right\} \quad . \quad . \quad . \quad (7)$$

In accordance with a theorem already proved,* if these equations have a solution in which x and y are proportional to $e^{\gamma t}$ when k is treated as constant, then they will have a solution proportional to $\exp \int \gamma dt$ when k is variable. The period equation is then

$$\begin{vmatrix} \gamma - 3n^2 + k\gamma & -2n\gamma \\ 2n\gamma + \frac{3}{2}nk & \gamma^2 + k\gamma \end{vmatrix} = 0,$$

* *Monthly Notices*, vol. lxxv., 1915, p. 577.

which reduces to

$$\gamma^2(\gamma^2 + n^2 + 2k\gamma) = 0 \quad . \quad . \quad . \quad (8)$$

The first factor, giving $\gamma = 0$, represents the motion in an orbit of slightly different major axis and epoch, and does not concern us. The other gives

$$\gamma = \pm \imath n - k \text{ to the first order} \quad . \quad . \quad . \quad (9)$$

Hence the amplitude of the oscillation about a circular orbit is proportional to $\exp \int -k\,dt$. Also there is no secular change in the mean distance to this order.

We thus see that the eccentricity of a planetary orbit is proportional to $\exp \int -k\,dt$, and it therefore diminishes with the time more rapidly than

$$\exp \int -\frac{1}{m}\frac{dm}{dt}\,dt, \text{ or } \frac{1}{m} \quad . \quad . \quad . \quad . \quad (10)$$

Hence if a planet picks up sufficient material to double its mass, its eccentricity is reduced to less than half its former amount.

Now the eccentricity of Jupiter's orbit at present is 0·048. Hence if its mass had ever been so small as $\frac{1}{20}$ of its present amount, its eccentricity must have been so large that the present approximation would break down. In other words, on this hypothesis of initially nearly circular orbits Jupiter can never have been smaller than Uranus is now. Planetary perturbations cannot account for the present eccentricity of Jupiter's orbit, as it was shown by Laplace that $\Sigma m a^{\frac{1}{2}} e^2$ for all the planets cannot be changed by their mutual perturbations, and most of this quantity is due to Jupiter.

This form of the planetesimal hypothesis might then be satisfactory if we could explain how the large planetary nuclei required by it originally came to exist, moving in very eccentric orbits, and with mean motions possibly not very different from their present values.

Further Note on Lagrange's Three Particles.

In my former paper* "On the Effect of a Resisting Medium on Lagrange's Three Particles" it was shown that the motion of the third particle is secularly stable, on the hypothesis that the velocity of the medium itself is constant through space. This hypothesis appears to agree with that made by Dr. See in the Capture Theory, as he considers the major axes of the orbits to be secularly diminished by the resistance, which seems to imply that he considers the medium to be at rest. In the case just considered, however, the velocity is a function of the position, so that if we take the same axes as were used in the previous paper, and (u, v, w) are the velocities of the medium at (x, y, z), we have $u = -ny$, $v = n\mathrm{R} - \frac{1}{2}nx$, $w = 0$. Further, n is in this case a constant.

* *Monthly Notices*, vol. lxxv., 1915, p. 580.

The velocities of the particle relative to fixed axes instantaneously coinciding with the movable ones are $\dot{x}-ny$, $\dot{y}+nx$, $\dot{z}$. The equations of motion in a non-dissipative system being

$$\left.\begin{aligned} \ddot{x}-2n\dot{y} &= ax+hy \\ \ddot{y}+2n\dot{x} &= bx+hy \\ \ddot{z} &= -n^2z \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (11)$$

we see that in this case the equations must be

$$\left.\begin{aligned} \ddot{x}-2n\dot{y} &= ax+hy-\kappa\dot{x} \\ \ddot{y}+2n\dot{x} &= hx+by-\kappa(\dot{y}+\tfrac{3}{2}nx) \\ \ddot{z} &= -n^2z-\kappa\dot{z} \end{aligned}\right\} \quad . \quad . \quad . \quad (12)$$

The last equation is independent of the other two, and hence the vibration in latitude dies down in such a way that its amplitude is proportional to $\exp -\frac{1}{2}\int\kappa\,dt$.

In the other two, suppose x and y proportional to $\exp\int\gamma\,dt$. Then γ is given by the equation

$$(\gamma^2-a)(\gamma^2-b)+4n^2\gamma^2-h^2+\kappa\{2\gamma^3-(a+b)\gamma+\tfrac{3}{2}n(2n\gamma+h)\}=0.$$

Now $a+b=3n^2$, and $ab-h^2=27n^4\mathrm{SJ}/8(\mathrm{S}+\mathrm{J})^2$.

Substituting these values we find,

$$\gamma^4+n^2\gamma^2+\frac{27n^4\mathrm{SJ}}{8(\mathrm{S}+\mathrm{J})^2}+\tfrac{3}{2}nh\kappa+2\kappa\gamma^3=0 \quad . \quad . \quad (13)$$

If κ were zero, the solutions would be $\gamma=\pm\iota\alpha$ and $\pm\iota\beta$, say.

Suppose then $\gamma-\iota\alpha$ to be small. The second approximation is found to be given by

$$\gamma-\iota\alpha=\frac{2\kappa\iota\alpha^3-\frac{3}{2}nh\kappa}{2\iota\alpha(\beta^2-\alpha^2)}.$$

The real part of γ is thus $\kappa\alpha^2/(\beta^2-\alpha^2)$ . . . . . (14)

Now let α correspond to the more rapid vibration about the steady motion. Then α is not very different from n, while β is much smaller. We see then that as the amplitude of the more rapid vibration is proportional to $\exp -\alpha^2\int\kappa dt/(\alpha^2-\beta^2)$ and that of the slower one is by symmetry proportional to $\exp\beta^2\int\kappa dt/(\alpha^2-\beta^2)$, it follows that the amplitude of the more rapid vibration decreases steadily, while that of the slower one increases with the time. The motion is therefore secularly unstable when the resisting medium is supposed to revolve.

The criticisms of the Planetesimal Hypothesis that have here been made rest entirely on the supposition that the planetesimals were solid bodies. If, instead of being solid particles, they were discrete molecules, then the nebula was indistinguishable from a gaseous one, possibly at a low temperature, and not rotating like a rigid body. Such a nebula would thus partake of features of

both the planetesimal nebula and the Laplacian nebula, and may combine advantages from both. A system of nuclei revolving in eccentric orbits within this would evolve to something not very different from the present solar system; it remains to be examined, however, whether these nuclei could have been small originally. If the relation that *em* decreases holds for this type of nebula as well as for the planetesimal one with nearly circular orbits, then the eccentricities of the present planetary orbits will be difficult to explain on this hypothesis, as they appear to be on almost any other. The further question of whether gravitational instability could occur in such a nebula as this requires to be discussed.

It scarcely seems likely that a system of this type would bear any resemblance to any known form of nebula. Viewed from α Centauri it would appear as a first-magnitude star, with a very faint, or even invisible, nebulosity round it; it would be enormously smaller than any nebula that is known to exist, and if any clue to the past history of the solar system is to be obtained from observations of bodies outside it, this hypothesis can be tested only by observations of nebulous stars, and not of nebulæ.

Appendix.

Note on a Type of Definite Integral.

In certain problems that have arisen earlier in this paper it has been found necessary to evaluate integrals of the type $\mathrm{I}\equiv\iiint\phi(\mathrm{S})(lx+my+nz+p)dx\,dy\,dz$, where S is the general quadratic function of x, y, and z, denoted by $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d$. The limits are defined by the condition that $lx+my+nz+p$ shall be positive; otherwise the integration is through all space. In all cases that have arisen the equation $\mathrm{S}=0$ represents an ellipsoid. The axes are supposed rectangular.

In the first place, the centre of this ellipsoid is given by the equations

$$ax'+hy'+gz'+u=0.$$
$$hx'+by'+fz'+v=0.$$
$$gx'+fy'+cz'+w=0.$$

Putting $x-x'=\mathrm{X}$, $y-y'=\mathrm{Y}$, $z-z'=\mathrm{Z}$, we have

$$\mathrm{S}\equiv a\mathrm{X}^2+b\mathrm{Y}^2+c\mathrm{Z}^2+2f\mathrm{YZ}+2g\mathrm{ZX}+2h\mathrm{XY}+\Delta/\mathrm{D},$$

where Δ denotes the discriminant of S, and capital letters (ABCFGHD) denote the minors of the corresponding small letters in Δ. Write $\Delta/\mathrm{D}=\delta$. There is no loss of generality if S is taken such that δ is always positive, so that S is positive inside the ellipsoid and negative outside.

Now let ξ denote the length of the perpendicular from the

point (x, y, z) to the plane through the centre parallel to the plane $lx+my+nz+p=0$. Then $\xi=l\mathrm{X}+m\mathrm{Y}+n\mathrm{Z}$.

If $-\xi_0=lx'+my'+nz'+p$, then

$$lx+my+nz+p=\xi-\xi_0.$$

ξ_0 is then given by the equation

$$\begin{vmatrix} l & m & n & p+\xi_0 \\ a & h & g & u \\ h & b & f & v \\ g & f & c & w \end{vmatrix}=0$$

Now take as a new set of axes any pair of conjugate diameters in the plane $\xi=0$ and the diameter conjugate to this plane. Let η and ζ be the distances measured along the axes in the plane $\xi=0$. The co-ordinate ξ is, however, the perpendicular from the point to the plane $\xi=0$, and not the distance measured along the ξ-axis. If we transform to co-ordinates (ξ, η, ζ), S becomes $\delta-\alpha\xi^2-\beta\eta^2-\gamma\zeta^2$, where α, β, and γ are constants. As the quadric is an ellipsoid, all have the same sign as δ, and there is no loss of generality in taking them all to be positive. Then the integral becomes

$$\int_{\xi_0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(\delta-\alpha\xi^2-\beta\eta^2-\gamma\zeta^2)(\xi-\xi_0)d\xi\,d\eta\,d\zeta\,\frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)}.$$

Putting $\xi=\mathrm{L}\sqrt{(\delta/\alpha)}$, $\eta=\mathrm{M}\sqrt{(\delta/\beta)}$, $\zeta=\mathrm{N}\sqrt{(\delta/\gamma)}$, we have

$$\mathrm{I}=\int_{\mathrm{L}_0}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi\{\delta(1-\mathrm{L}^2-\mathrm{M}^2-\mathrm{N}^2)\}(\mathrm{L}-\mathrm{L}_0)d\mathrm{L}\,d\mathrm{M}\,d\mathrm{N}\frac{\delta^2}{\alpha(\beta\gamma)^{\frac{1}{2}}}\frac{\partial(x,y,z)}{\partial(\xi,\eta,\zeta)}$$

where $\xi_0=\mathrm{L}_0\sqrt{(\delta/\alpha)}$.

The numerical factor must now be evaluated. To do this, consider the plane $l\mathrm{X}+m\mathrm{Y}+n\mathrm{Z}+q=0$. This touches the quadric $\mathrm{S}=0$ if $(bc-f^2)l^2+(ca-g^2)m^2+(ab-h^2)n^2+2(gh-af)mn+2(hf-bg)nl+2(fg-ch)lm+\mathrm{D}q^2/\delta=0$.

Now the appropriate values of q must be $\pm(\delta/\alpha)^{\frac{1}{2}}$. Hence

$$-\frac{\mathrm{D}}{\alpha}=(bc-f^2)l^2+(ca-h^2)m^2+(ab-h^2)n^2+2(gh-af)mn+2(hf-bg)nl+2(fg-ch)lm.$$

This determines α.

Now an element of volume $=dx\,dy\,dz$

and also $\qquad\qquad=d\xi\,d\eta\,d\zeta\sin\psi,$

where ψ is the angle between the conjugate diameters chosen for axes of η and ζ. Hence

$$\frac{\delta(x,y,z)}{\delta(\xi,\eta,\zeta)}=\sin\psi.$$

Now $\dfrac{\delta}{\sqrt{(\beta\gamma)}}\sin\psi$ = area of the parallelogram with the semi-conjugate diameters as adjacent sides,
= $\frac{1}{8}$ of the volume of a parallelopiped circumscribing the ellipsoid $\div\frac{1}{2}$ the thickness of this parallelopiped perpendicular to the plane $\xi=0$.

But $\frac{1}{8}$ of the volume of this parallelopiped $=\alpha_1\beta_1\gamma_1$, where α_1, β_1, and γ_1 are the principal semi-axes of the ellipsoid. We find at once that $\alpha_1\beta_1\gamma_1=(-\delta^3/D)^{\frac{1}{2}}$. Also the divisor $=(\delta/a)^{\frac{1}{2}}$.

Hence we find on substitution that

$$\frac{\delta^2}{a(\beta\gamma)^{\frac{1}{2}}}\frac{\delta(x,y,z)}{\delta(\xi,\eta,\zeta)}=\frac{\delta^2}{(-aD)^{\frac{1}{2}}}.$$

Finally, by a transformation of the variables M and N, the integral is reduced to the form

$$I=\frac{2\pi\delta^2}{(-aD)^{\frac{1}{2}}}\int_{L_0}^{\infty}\int_{0}^{\infty}\phi\{\delta(1-L^2-P^2)\}(L-L_0)P\,dL\,dP.$$

By putting $P^2=R^2-L^2$, it is possible to reduce this to a single integral, equal when L_0 is positive to

$$\frac{\pi\delta^2}{(-aD)^{\frac{1}{2}}}\int_{L_0}^{\infty}(R-L_0)^2\phi\{\delta(1-R^2)\}R\,dR.$$

When L_0 is negative, equal to $-H_0$, the limits become more complicated, and the integral is

$$\frac{\pi\delta^2}{(-aD)^{\frac{1}{2}}}\left[\int_{H_0}^{\infty}(R+H_0)^2\phi\{\delta(1-R^2)\}R\,dR-4\int_{0}^{H_0}L_0R^2\phi\{\delta(1-R^2)\}dR\right].$$

In practice, however, it appears to be often simpler to evaluate the double integral directly by integrating with regard to P than to use this single integral.

In the general case it is not possible to make any further progress without specifying the form of ϕ, which at present is quite arbitrary except that it must be such that no logical difficulties arise in the transformations of the infinite integrals. In the two particular cases that have arisen the integration can be completed.

(1) Let $\phi(S)=\rho\exp S$.

Then it is evident that, on account of the rapidity with which $\exp S$ tends to zero when any of X, Y, and Z becomes large, all the previous transformations are justified in this case. Hence

$$I=\frac{\rho\pi\delta}{\sqrt{(-aD)}}\left\{\frac{1}{2\delta}e^{-\delta L_0^2}-\frac{L_0}{\sqrt{\delta}}\frac{\pi^{\frac{1}{2}}}{2}(1-\mathrm{Erf}L_0\sqrt{\delta})\right\}.$$

(2) Let $S=\delta-aX^2-bY^2-cZ^2$, and $\phi(S)=\rho S^k$ when S is positive, and zero when S is negative.

Then $$\frac{1}{a}=\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{e}.$$

$$I=\frac{\pi\rho\delta^{k+2}}{(k+1)\sqrt{(abca)}}\left[\frac{(1-L_0^2)^{k+2}}{2(k+2)}-L_0\int_0^{\theta_0}\sin^{2k+3}\theta\,d\theta\right]$$

where $\cos\theta_0=L_0$ and θ_0 is between 0 and π.

The Secular Perturbations of the Four Inner Planets.
By Harold Jeffreys, B.A., M.Sc.

It was shown by Newcomb* that the observed secular variations in the orbits of the inner planets differ slightly from those calculated from the law of gravitation, the best available values of the masses being adopted. The principal discrepancies are in the movements of the perihelion of Mercury and the node of Venus, but certain others also have notable values. Two important attempts have been made to account for the differences. The first was by Seeliger,† who assumed that about the sun are two superposed oblate homogeneous ellipsoids of gravitating matter. In addition he assumed that the axes of reference, which are fixed relative to the mean of the stars observed, are actually rotating in the plane of the ecliptic. By means of a method of least squares he found the best values of the densities of the ellipsoids and the rotation; all the residuals, with the exception of those in the eccentricity of Mercury and Mars, the explanation of which does not seem to have been attempted by anyone, were less than their mean errors. It thus appeared very satisfactory. Nevertheless, it seems to require serious modification, on account of the doubtful character of some of the physical hypotheses involved.

(*a*) The rotation found appears too great to be possible. De Sitter‡ found, however, that it is not necessary to the theory, for on rejecting it he obtained a satisfactory representation by means of the ellipsoids alone.

(*b*) The density of the outer ellipsoid, which extends outside of the orbit of the earth, was found by De Sitter to be $3{\cdot}7\times10^{-15}$, and by Seeliger to be $3{\cdot}1\times10^{-15}$ of that of the sun. It will be shown later that this density, small as it seems, is greater than can actually be allowed.

On the other hand, Einstein§ has recently, by means of a form of the Principle of Relativity, given an excellent representation of the perihelion of Mercury and a fair one of that of Mars; but

* *Astronomical Constants*, p. 109.

† *Sitzungsber. K. Bayer. Akad., Math.-Phys. Klasse*, 1906, p. 595.

‡ *Observatory*, vol. xxxvi., 1913, p. 296. This also gives an account of the earlier work on the subject.

§ *Annalen der Physik*, Bd. xlix., 1916, p. 769.

naturally no hypothesis of this type can affect the position of the plane of an orbit, so that the empirical part of the motion of the node of Venus remains unexplained. For this reason it appears desirable to try to find the distribution of gravitating matter that will best account for the changes in the inclinations and nodes, for it seems impossible to account for them otherwise; this can then be used to determine the motions of the perihelia, to ascertain how much of these can be accounted for by such means. The disturbing matter must be assumed to be close to the sun, as the density near the orbit of Mercury and outside it will be shown to be so small that it can have no perceptible effect. When this is done, the distribution that gives the best approximation to the movements of the planes of the orbits is capable of causing an advance of the perihelion of Mercury $1\frac{1}{2}$ times as great as the observed excess.

The Density of the Matter causing the Zodiacal Light.— Seeliger adopted the special dimensions that he chose for his outer ellipsoid because the observed phenomenon of the zodiacal light indicates the presence of matter as far out as he indicated, and in particular between the orbits of Mercury and the earth. Within the orbit of Mercury the observations are not so decisive, as they must be made so soon after sunset or before sunrise that matter there is practically invisible. Presumably, however, the density increases rapidly towards the sun, as Seeliger supposed. The nature of the substance causing the light is uncertain, but it may be supposed either to be composed of particles of meteoric type, large compared with the wave-length of light, or of very fine dust or gaseous matter, the particles being small compared with the wave-length. The treatment of the problem is different in the two cases.

In the first case, let the number of particles per unit volume be ν, and their mean diameter σ. Then it was shown * from the observed luminosity of the zodiacal light that the value of $\nu\sigma^2$ is about $3{\cdot}5 \times 10^{-22} \div 1$ cm. for the neighbourhood of the orbit of Venus. The determination was not an accurate one, but the true value is not likely to be more than 10 times this or less than $\frac{1}{10}$ of it. The density is $\frac{1}{6}\pi\rho\nu\sigma^3$, where ρ is the density of a single particle. Now, putting this equal to De Sitter's density of $3{\cdot}7 \times 10^{-15}$ of that of the sun, we see that the above value of $\nu\sigma^2$ would require the diameters of the particles to be about 10^7 cm. This value is so great that the particles would appear as small separate planets instead of as a continuous sheet. Any size that would be acceptable, taken in conjunction with the above value of $\nu\sigma^2$, would make the density so small that it is quite evident that the large particles that may help to cause the zodiacal light must be so few that their effect on the orbits of the planets must be hundreds of times less than was required of the outer ellipsoid.

If the particles are small compared with the wave-length, the

* P. 94.

problem is considerably altered. The light is no longer diffusely reflected from the surface, but is scattered instead.* The intensity is now proportional to $\nu\sigma^6$ instead of $\nu\sigma^2$, and a comparison of the theory with observation may be derived from the scattering of light by the earth's atmosphere. If the diameters of the particles were the same in the two cases, the amounts of light received from equal small angles, first, of the blue sky in daytime, and second, of the zodiacal light, would be proportional to the masses of columns of equal cross-section through the two structures considered. This ratio, on Seeliger's hypothesis, would be $1{\cdot}2 \times 10^4$. The observed ratio of the luminosity of the sky to that of an equal area of the sun is given by L. V. King † to be 80×10^{-8} at the Mount Wilson level, which was considered sufficiently high above sea-level for dust to be of no importance. The intensity of the light from the sky is therefore 8×10^6 of that of the zodiacal light, so that the latter is of the order of 600 times fainter than would be expected on the hypothesis that both it and the light from the sky are due to molecular scattering. Lord Kelvin's estimate ‡ gives values ranging from 4×10^{-7} to $2{\cdot}6 \times 10^{-7}$ for the ratio of sky to sun, which would reduce the above ratio to about 200. If the zodiacal light were caused by particles much larger than the molecules in the air, the discrepancy would be still greater, as $\nu\sigma^6$ would be greater for the same value of $\nu\sigma^3$. It seems very improbable then that the matter causing the zodiacal light can have a density in any way comparable with that of the outer ellipsoid, and its effect on the motions of the planets may accordingly be neglected.

The Secular Motions of a Planet consequent on an Oblate Distribution close to the Sun.—The hypothesis here adopted to give the distribution of the disturbing matter is a modification of that of Seeliger, so as to make it consistent with photometric observations. Any important perturbing mass must then be far inside the orbit of Mercury. The potential can then be expressed in terms of a series of spherical solid harmonics of the form

$$V = f\frac{M}{r} + \frac{S_2}{r^3} + \frac{S_3}{r^4} + \frac{S_4}{r^5} + \ldots,$$

where S_2, S_3, S_4, etc., are surface harmonics of the second, third, fourth, and higher orders respectively. Further, it can be seen that as it is highly improbable that a mass of this type is not rotating about an axis through the centre of the sun, the harmonics involved must be zonal. It can also be shown without much difficulty that odd harmonics can give rise to no secular terms in the motion of a planet that moves near the equatorial plane of the distribution. S_3 may therefore be omitted. In this investigation S_4 will not be considered, as its effect will be small on every planet except Mercury; if later observations show this to be

* R. W. Wood, *Physical Optics*, pp. 478–480.
† *Phil. Trans.*, 212A, 1912–13, p. 411.
‡ *Baltimore Lectures*, p. 314.

desirable, however, this term will have to be included and determined with the rest.

Let now the sum of the masses of the sun and the perturbing matter be M,
the moments of inertia of the perturbing mass about its polar and equatorial axes be C and A,
the colatitude of any point, measured from the polar axis, be θ,

and put $a = \frac{1}{2}(C - A)$.

a is positive when the distribution is oblate.

Then

$$V = f\frac{M}{r} + f\frac{a(1 - 3\cos^2\theta)}{r^3}.$$

If now x, y, z be the Cartesian co-ordinates of a point, and (l, m, n) the direction cosines of the polar axis, the disturbing function is

$$R = far^{-5}\{r^2 - 3(lx + my + nz)^2\}.$$

Now, in the usual notation of the planetary theory,

$$r = a(1 - e^2)/(1 + e\cos w),$$
$$x = r\{\cos\Omega\cos(\nu - \Omega) - \sin\Omega\sin(\nu - \Omega)\cos i\},$$
$$y = r\{\sin\Omega\cos(\nu - \Omega) + \cos\Omega\sin(\nu - \Omega)\cos i\},$$
$$z = r\sin(\nu - \Omega)\sin i.$$

If ω be the mean motion of the planet, so that $\omega = (\mu/a^3)^{\frac{1}{2}}$, the disturbing function is periodic in a time $2\pi/\omega$. The secular perturbations depend only on the secular part of R, which is the average of R taken with regard to the time. If this be S, we have

$$S = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} R\,dt,$$

$$= \frac{fa}{a^3(1 - e^2)^{\frac{3}{2}}}\left[1 - \tfrac{3}{2}(l\cos\Omega + m\sin\Omega)^2 - \tfrac{3}{2}(-l\sin\Omega\cos i + m\cos\Omega\cos i + n\sin i)^2\right].$$

The perturbations are now given by the equations *

$$\frac{da}{dt} = \frac{2}{\omega a}\frac{\partial S}{\partial\epsilon},$$

$$\sin i\frac{d\Omega}{dt} = \frac{1}{\omega a^2\sqrt{(1 - e^2)}}\frac{\partial S}{\partial i},$$

$$\frac{d\varpi}{dt} = \frac{\sqrt{(1 - e^2)}}{\omega a^2}\frac{\partial S}{e\partial e} + \frac{\tan\frac{1}{2}i}{\omega a^2\sqrt{(1 - e^2)}}\frac{\partial S}{\partial i},$$

$$\frac{de}{dt} = -\frac{\sqrt{(1 - e^2)}}{\omega a^2 e}\frac{\partial S}{\partial\varpi} - \sqrt{(1 - e^2)}\frac{1 - \sqrt{(1 - e^2)}}{\omega a^2 e}\frac{\partial S}{\partial e},$$

$$\frac{di}{dt} = -\frac{1}{\omega a^2\sqrt{(1 - e^2)}\sin i}\frac{\partial S}{\partial\Omega} - \frac{\tan\frac{1}{2}i}{\omega a^2\sqrt{(1 - e^2)}}\left(\frac{\partial S}{\partial\varpi} + \frac{\partial S}{\partial\epsilon}\right),$$

$$\frac{d\epsilon}{dt} = -\frac{2}{\omega a}\frac{\partial S}{\partial a} + \frac{\tan\frac{1}{2}i}{\omega a^2\sqrt{(1 - e^2)}}\frac{\partial S}{\partial i} + \sqrt{(1 - e^2)}\frac{1 - \sqrt{(1 - e^2)}}{\omega a^2 e}\frac{\partial S}{\partial e}.$$

* Tisserand, *Mécanique Céleste*, t. 1, p. 369.

We see at once that $\frac{da}{dt}$ and $\frac{de}{dt}$ are zero. In the other variations, the parts depending on the squares of the small quantities i, l, m can be omitted, except in the case of the perihelion of Mercury. In this case squares will be retained, but no higher powers.

$$\sin i \frac{d\Omega}{dt} = -\frac{3fa}{\omega a^5(1-e^2)^2}(\sin i - l\sin\Omega + m\cos\Omega),$$

$$\frac{di}{dt} = -\frac{3fa}{\omega a^5(1-e^2)^2}(l\cos\Omega + m\sin\Omega),$$

$$\frac{d\varpi}{dt} = \frac{3fa}{\omega a^5(1-e^2)^2}\left[1 - \tfrac{3}{2}(l-\sin i\sin\Omega)^2 - \tfrac{3}{2}(m+\sin i\cos\Omega)^2 - \tfrac{1}{2}\sin i(\sin i - l\sin\Omega + m\cos\Omega)\right].$$

Now choose the units so that the unit of length is the earth's mean distance from the sun, that of time is such that the earth's mean motion is unity, and the sum of the masses of the sun and earth is the unit of mass. Then $f=1$, and for any planet $\omega = a^{-\frac{3}{2}}$.

The variations found from the last equations will then be expressed in radians per unit of time, whereas they are recorded by Newcomb in seconds of arc per century, the former unit being $1{\cdot}296\times10^8$ times the latter. It will therefore be found convenient to introduce a new quantity $\beta = 1{\cdot}296\times10^8 a$.

The following table gives the elements of the orbits of the four inner planets, with their unexplained secular variations in a century and their mean errors :—

Planet.	a.	e.	$\frac{3}{\omega a^5(1-e^2)^2}$.	Ω.	i.	$\sin i\delta\Omega$.	δi.	δe.	$e\delta\varpi$.
				° ′ ″	° ′ ″				
Mercury	·3871	·206	72·01	47 8 45	7 0 10	0″·61±0″·51	0″·38±0″·80	−0″·88±0″·50	8″·48±0″·43
Venus	·7233	·007	10·46	75 46 47	3 23 27	0·60±0·17	0·38±0·33	0·21±0·31	−0·05±0·25
Earth	1·0000	·017	3·002	...	0 0 0	...	−0″·22*±0″·27	0″·02±0″·10	0″·10±0″·13
Mars	1·5237	·093	·5557	48 47 9	1 15 1	0·03±0·22	−0″·01±0″·20	0″·29±0″·27	0″·75±0″·34

Let us first consider the motions of the planes of the orbits separately.

The secular changes in i and Ω give then the following conditional equations to determine the unknowns β, l, m :—

$$\left.\begin{aligned}(-49{\cdot}9l - 52{\cdot}7m)\beta &= 0{\cdot}38\pm0{\cdot}80\\ (+52{\cdot}7l - 49\,9m - 8{\cdot}79)\beta &= 0{\cdot}61\pm0{\cdot}51\end{aligned}\right\}\text{ from Mercury.}$$

$$\left.\begin{aligned}(-2{\cdot}57l - 10{\cdot}14m)\beta &= 0{\cdot}38\pm0{\cdot}33\\ (10{\cdot}14l - 2{\cdot}57m - 0{\cdot}62)\beta &= 0{\cdot}60\pm0{\cdot}17\end{aligned}\right\}\text{ from Venus.}$$

$$\left.\begin{aligned}(-{\cdot}363l - {\cdot}417m)\beta &= -0{\cdot}01\pm0{\cdot}20\\ ({\cdot}417l - {\cdot}363m - {\cdot}0121)\beta &= 0{\cdot}03\pm0{\cdot}22\end{aligned}\right\}\text{ from Mars.}$$

* This entry is not δi, but $\delta\epsilon$.

The normal equations derived from these, weights being assigned according to the mean errors, are—

$$(36{\cdot}22l - 13{\cdot}17m - 4{\cdot}095)\beta - 0{\cdot}594 = 0.$$
$$(-13{\cdot}17l + 29{\cdot}30m + 3{\cdot}473)\beta + 0{\cdot}466 = 0.$$
$$(-4{\cdot}095l + 3{\cdot}473m + 0{\cdot}640)\beta + 0{\cdot}069 = 0.$$

From these we find $l = +0{\cdot}09503$; $m = -0{\cdot}09005$; $\beta = {\cdot}915$. Thus the equatorial plane of the disturbing matter is inclined at $7^\circ\ 31'$ to the ecliptic, and the longitude of its ascending node is $46^\circ\ 33'$. It therefore nearly coincides with the orbit of Mercury.

The values of the perturbations caused in a century by the distribution thus found are as follows, expressed in seconds of arc:—

Planet.	sin $i\delta\Omega$.	δi.	$e\delta\bar{\omega}$.
Mercury	0·64	0·00	13·6
Venus	0·52	0·61	0·06
Earth	...	...	0·05
Mars	0·05	0·00	0·05

The distribution found thus accounts for all the unexplained secular motions with the exceptions of those of the perihelia of Mercury and Mars and the variations in the eccentricities of the same planets. The residuals in all other cases are less than the mean errors. On the other hand, it is seen that the motions of the orbital planes of the planets actually imply a secular movement of the perihelion of Mercury half as large again as the unexplained part already known to exist.

Next, consider the motions of the orbital planes and of the perihelia together in order to determine the best values of l, m, and β. In this case the equations for $ed\bar{\omega}/dt$ will give four conditional equations in addition to those already used.

$(14{\cdot}72 + {\cdot}660l - {\cdot}613)\beta = 8^\circ{\cdot}48 \pm 0^\circ{\cdot}43$ ·
(squares of $l - \sin i \sin\Omega$ and $m + \sin i \cos\Omega$ being omitted)

$${\cdot}07\beta = -0^\circ{\cdot}05 \pm 0^\circ{\cdot}25$$
$${\cdot}05\beta = 0^\circ{\cdot}10 \pm 0^\circ{\cdot}13$$
$${\cdot}05\beta = 0^\circ{\cdot}75 \pm 0^\circ{\cdot}34$$

The normal equations now found are—

$$(36{\cdot}22l - 13{\cdot}17m - 3{\cdot}99)\beta - 0{\cdot}654 = 0,$$
$$(-13{\cdot}17l + 29{\cdot}30m + 3{\cdot}375)\beta + 0{\cdot}522 = 0,$$
$$(-3{\cdot}990l + 3{\cdot}375m + 2{\cdot}984)\beta - 1{\cdot}282 = 0,$$

whence

$$l = +0{\cdot}1057;\ m = -0{\cdot}0986;\ \beta = 0{\cdot}575.$$

The perturbations caused by this distribution, with the same units as before, are given in the following table :—

Planet.	$\sin i\delta\Omega$.	δi.	$e\delta\varpi$.
Mercury	0·94	0·00	8·53
Venus	0·41	0·42	0·04
Earth	...	...	0·03
Mars	0·04	0·00	0·03

The secular variations are thus all accounted for within their mean errors except the motions of the perihelion of Mars and the node of Venus. In the latter case, however, the residual is only just greater than the mean error. The equatorial plane of the perturbing matter is inclined at 8° 19′ to the ecliptic, and the longitude of its ascending node is 47° 0′. Thus it is sufficiently near to the plane of the orbit of Mercury for the quantity $(l-\sin i \sin\Omega)^2+(m+\sin i\cos\Omega)^2$ to be neglected. The value found for β shows that $\alpha=4{\cdot}43\times10^{-9}$. If the disturbing potential was to be attributed to an ellipticity of the sun, the requisite value of this would be $9{\cdot}93\times10^{-3}$ (the sun being assumed homogeneous), which is much too great to be possible. The matter must therefore be outside of the sun, though its diameter must be much less than that of the orbit of Mercury. A somewhat better representation could be obtained by including a fourth harmonic, but in view of the size of the other unexplained discrepancies this scarcely seems worth doing at present.

If then Einstein's hypothesis be accepted, any perturbing matter that may exist must be such as to produce considerable effects on the positions of the planes of the orbits, without affecting the much more sensitive perihelion of Mercury. This can only occur if a fourth harmonic is included in the potential, and so accurately adjusted that its effect on the perihelion of Mercury just balances that of the second harmonic. In the figure of the disturbing mass the fourth harmonic must then be very much more prominent than the second harmonic, which appears unlikely. It appears therefore that on the hypothesis of Einstein it will be found very difficult to find a plausible explanation of the motions of the planes of the orbits, and in particular of the node of Venus.

The Irregular Variable RS Camelopardalis.

By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

1. The valuable paper by C. Martin and H. C. Plummer in the May number of the *Mon. Not.* (p. 612) gives not only a fine series of observations of this star extending over more than four years, but a careful analysis of them resulting in an apparent solution of the problem of irregular variability, possibly extending to other

cases. The solution is, moreover, put into elegant shape; and it is an ungracious task to challenge the interpretation of it (nothing here said challenges the analysis). Indeed, the following criticism did not shape itself until the paper had been studied with much pleasure and approval, and it represents a considerable disappointment to the writer; for it seemed on first reading the paper as though a satisfactory case had at last been found of a phenomenon which has hitherto proved very elusive.

2. It is well known that Argelander gave a formula for the oscillations of the maxima of variable stars of the form

$$\text{Julian Date} + 255\text{E} + 15 \sin (9^\circ\text{E} + 230^\circ)$$

where 255 days is the mean period, but the maxima are presumed to oscillate 15 days either way from their mean position in a long cycle completed in $360^\circ/9^\circ = 40$ periods of 255 days, say 30 years. Now although formulæ of this kind have been fitted by Chandler and others to the observations of variables extending over a limited period, *no case* has (so far as I am aware) yet been found in which the periodic term has satisfied a complete cycle of its own changes, and brought back the maxima to their original sequence. On the other hand, there have been various cases of breakdown when such a formula *might* have come round again, but has failed to do so: so that a quite different hypothesis of several abrupt changes of period has been adopted for more than one such case, after the Argelander-Chandler formula had clearly broken down.

3. Now Messrs. Martin and Plummer seemed at last to have found a satisfactory case with a complete oscillation of the maximum: they do not express it in that form, but it can be so expressed. Their mode of expression contemplates "a simple oscillation in the period 88·7 days, with slowly changing amplitude and phase." We shall omit consideration at present of the amplitude, and consider only the phase ψ for which they give the equation (p. 627):

$$\tan \psi = \frac{\sin (0^\circ{\cdot}32t' - 63^\circ)}{3 + 6 \cos (0^\circ{\cdot}32t' - 63^\circ)} = \frac{\sin (\theta - \phi)}{3 + 6 \cos (\theta - \phi)}$$

the second form being virtually used on the previous page where the variation in 88·7 days is represented by the angle $(\theta + \phi)$.

4. In this form the phase ψ consists partly of a secular and partly of a periodic term: for when

$(\theta - \phi)$	=	0°	120°	180°	240°	360° etc.
ψ	=	0°	90°	180°	270°	360° etc.

so that the secular part of ψ is clearly $(\theta - \phi)$ or χ say.

To get the periodic part ω write

$$\psi = \chi - \omega.$$

Then

$$\tan \omega = \tan (\chi - \psi) = \frac{\sin \chi (3 + 5 \cos \chi)}{1 + 3 \cos \chi + 5 \cos^2 \chi},$$

and since the denominator can never vanish, ω oscillates between limits less than $\pm 90°$. If, then, we add this secular variation of the phase to the argument, we are left with an oscillating phase such as could be represented by an Argelander-Chandler formula. Moreover, the oscillation of the phase takes place in the period of χ or $\theta - \phi$ or $(0°{\cdot}32t' - 63°)$, which is 1125 days, well covered by the extent of the observations. The beautiful accordance of these with the suggested formula is shown by the details of Table VIII. in the paper. Hence at last we seem to have an oscillation of the maximum, satisfactorily confirmed: it is really heart-breaking to have to raise doubts as to the reality of this long-delayed success. But previous personal experience of a noteworthy pitfall leaves no option.

5. Let us first recall the main point here criticised. On p. 626 the authors obtain the elegant expression

$$2a \sin 2\theta + (a+b) \sin (\theta + \phi) + 2b \sin 2\phi$$

for the variation, in which both the coefficients $2a$, $a+b$, $2b$, and the arguments 2θ, $\theta+\phi$, 2ϕ are in A.P. Attention is here entirely directed to the significance of the latter A.P. The relation between the coefficients may very possibly be of valuable significance: this point is not at present under examination: we confine ourselves to the inquiry whether the fact that a great part of the variation can be represented by three periodic terms with arguments in A.P. gives us any important information—for instance, any information which can be used for prediction—and the point of view here taken is that it does *not*; that the variation *may* be quite irregular (though with an approximate average period) and still be expressible in this form.

6. It will be convenient to give some notion of what is meant by an *irregular* variable, with an approximate average period.

The calendar months in our year have either 28, 29, 30, or 31 days; but their lengths are assigned in a definite way, and this is a case of regular, though multiple, periodicity. But suppose that when any month was starting, a die were to be thrown to determine its length. Let the usual numbers 1 to 6 mean that the length shall be 28, 29, . . . 33 days. Then the *average* length would remain much as at present but the month would become an irregular variable; i.e. *its actual length could not be predicted.*

7. Now let a series of observations of the lengths of such months be made for a year. The average period of something like $30\frac{1}{2}$ days would be easily apparent: but it would also be clear that this was not the whole story. Periodogram analysis would give us a maximum "intensity" at about $30\frac{1}{2}$ days, and the idea might occur to us, "Let us remove this term and see what is left." Call the term so removed $B \sin (\theta + \phi)$. Then when we removed it, we should find a term $A \sin 2\theta$ suggesting itself: it would be natural to remove that, and we should find then a term $C \sin 2\phi$ very clearly marked: and we should find the greater

part of the variation represented by these three terms. We should be tempted (the present writer has easily yielded to the temptation in more than one instance) to think that we had found out something definite about the law of variation. But the neat mathematical expression is applicable to all cases: it arises naturally just as much in the case of absolutely irregular variability above sketched as in the case of sun-spots, where several actual and determinable periods are superposed. So far this is a mere assertion: it remains to justify it.

8. When, in the instance before us, we determine the term B sin $(\theta+\phi)$ we are not finding an actual periodicity of $30\frac{1}{2}$ days: indeed, there is not any such: the nearest actual periodicities are 30 and 31 days. We shall include in B sin $(\theta+\phi)$ *contributions* from these two periodicities, but not the whole of them. We shall also include contributions from the 29-day and 32-day months, though not so large, owing to the greater deviation from the adopted period of $30\frac{1}{2}$ days. But as we leave the adopted period further we come to periodicities which are scarcely included at all, because at one time of the year-of-observation they are in accord, at another time in discord, with the average $30\frac{1}{2}$-day term. For 12 periods of 30·5 days make 366 days, and

$$366/11 = 33{\cdot}3\,, \qquad 366/12 = 30{\cdot}5\,, \qquad 366/13 = 28{\cdot}2.$$

Hence a 33-day period "interferes" with a $30\frac{1}{2}$-day in something like a year; and so does a 28-day. When the term B sin $(\theta+\phi)$ is determined and removed, two periodicities on opposite sides of it will be *untouched*, and will consequently become prominent. One or other of these will probably catch the eye more than the other—in § 7 we have assumed that it was the A sin 2θ term: and up to this point the notation used is quite general, for neither ϕ nor θ has been defined separately: but once we have determined A sin 2θ + B sin $(\theta+\phi)$, then the chief term left will be C sin 2ϕ, so that the arguments are in A.P.

9. It is worth going over the selection of θ and ϕ more explicitly. When we deliberately analyse the observations into a Fourier sequence of exact submultiples of the whole time T (in this case one year), then the relations of θ and ϕ are quite definite. But the existence of three terms with arguments in A.P. is a little more general than this. We may start as above with an argument $(\theta+\phi)$ selected by inspection, which gives us two arbitrary variables, θ and ϕ: then we may select, also by inspection perhaps, another argument 2θ. But the difference $(\theta+\phi-2\theta) = (\phi-\theta)$ is not really arbitrary: it is such that the $(\phi+\theta)$ term runs through its cycle *just once more* than the 2θ term during T the time available for discussion: and a term which similarly runs through its cycle just once less must therefore have argument $(\theta+\phi)+(\phi-\theta) = 2\phi$.

10. Besides these terms in 2θ and 2ϕ there are others which run through their cycle exactly twice, three times, etc., more or less than the adopted $(\theta+\phi)$ term with which we started, and which will therefore be quite untouched when we remove $(\theta+\phi)$.

The arguments for the next pair will be

$$(\theta+\phi) \pm 2(\phi-\theta) = 3\phi-\theta \text{ and } 3\theta-\phi,$$

and so on. Whether these will have sensible coefficients depends upon how near they lie to the periods which actually occur. In the above example the actual period of 33 days lies *between* $(\theta+\phi)$ which has been adopted as of 30·5 days, and 2θ which has been assumed to be 33·3 days. Hence it can be divided into components the chief of which has 33·3-day period, the next has 30·5, and but little is left for anything outside these. But 28 days lies *outside* the corresponding space between 30·5 and 28·2; so that it will have a sensible component on the small side of 28·2 days. In this actual example we should probably find that three terms would not be sufficient to express the main features of the observations: but if we either shorten the whole period of observation, or suppose the die thrown has only four numbered faces, restricting the possible periods to 29, 30, 31, or 32 days, we shall find three terms enough. The mode of distribution of the actual periodicities among the constituents of a Fourier sequence is given in *Monthly Notices, R.A.S.*, lxxiii. p. 718. and it seems probable (though the point has not yet been examined) that it is applicable also in the case when we start (as above) with an arbitrary term instead of an exact submultiple of the time of observation.

11. To sum up: the point here made is that *any* case of variation with an approximate average period, however irregularly the variation may be distributed about that period, can be represented by three terms with *arguments in A.P.*, provided the whole period of observation is short enough. How short it should be depends on the limits of departure from average period. Whatever these limits may be (provided they remain the same) then at first three terms in A.P. will suffice: but as the time of observation is extended the common difference of the A.P. will diminish and five terms will ultimately be required, still in A.P.: then seven, and so on. Hence there is no *necessary* significance in the relation of the arguments of the three terms given by Martin and Plummer on p. 626, *loc. cit.* The relation of the *coefficients* may be significant: that point is not here raised.

12. In the course of meditation on the particular problem here presented another point of a more general kind was suggested, which seems worthy of notice. The variation of RS Camelopardalis is represented by Martin and Plummer as a "simple oscillation in a period 88·7 days with a slowly changing amplitude and phase" (p. 626). The variations of the phase ψ are, as already quoted in § 3 of this paper, given by

$$\tan\psi = \frac{\sin(\theta-\phi)}{3+6\cos(\theta-\phi)},$$

and it is there shown that we may write

$$\psi - (\theta-\phi) - \omega$$

where ω is periodic. Hence the secular part of ψ is $(\theta-\phi)$. But ψ is the variable phase of the adopted period $(\theta+\phi)$. Should we not therefore include the secular part of ψ and say that the true period is $(\theta+\phi)+(\theta-\phi)=2\theta$? If so, then we just undo the work which Martin and Plummer go through in *combining* the three terms

$$0{\cdot}172 \sin 2\theta + \quad 0{\cdot}147 \sin(\theta+\phi) + \quad 0{\cdot}122 \sin 2\phi.$$

They combine them, by a pretty piece of analysis, to form a single term in $\theta+\phi$ with varying phase ψ; and we insist on restoring the secular change in ψ, thus obtaining a term in 2θ.

13. On full consideration, it is here urged that the latter is the better and truer view: that 2θ is the best measure of the periodicity, because the term in 2θ has the largest coefficient: and in general that *when several periodic terms co-exist, the terms with the largest coefficient settles the period*, no analytical combination of terms being really effective.

Thus the expression

$$(1+n^2)\cos 2pt + (1-n^2)\cos 2qt$$

can be written analytically

$$2\cos(p-q)t\cos(p+q)t + 2n^2 \sin(p-q)t \sin(p+q)t$$

which may be regarded (following Martin and Plummer) as having an argument $(p+q)t$ with slowly varying amplitude and phase, viz.

$$\text{amplitude} = \{4\cos^2(p-q)t + 4n^2\sin^2(p-q)t\}^{\frac{1}{2}}$$

and phase ψ given by

$$\tan\psi = n^2 \tan(p-q)t.$$

But it is clear that ψ contains the secular term $(p-q)t$, and if we remove this by adding it to the argument we get as before $2pt$, which is the term with the largest coefficient $(1+n^2)$

14. The impulse to combine two terms of different periods comes without doubt from the theory of "interference" in physical optics, where one coefficient is in general *equal* to the other, *i.e.* where $n^2=0$. In this particular case we cannot pick out one term as having a larger coefficient: and the sum of the terms is usually represented, from considerations of symmetry, as a periodic term $(p+q)t$, and a coefficient which is alternately positive and negative. But this is, after all, not a satisfactory way of regarding a periodicity. It would be more satisfactory to keep the coefficient always of one sign, and this we can do by choosing *either* of the arguments $2pt$ or $2qt$ instead of the symmetrical $(p+q)t$. We cannot prevent the coefficient vanishing, but we can prevent its changing sign.

15. A safer guide to the combination of two periodicities is afforded by the circulation of two planets round the Sun. In this

case it is always the outer planet, *i.e.* the one of greater radius or coefficient, which settles the period. The period of Jupiter seen from the Earth is that of Jupiter, with a little oscillation due to the Earth. The period of Venus seen from the Earth is that of the Earth, with an oscillation due to Venus. The case of equal coefficients (and different periods) would correspond to that of two planets circulating at the same distance from the Sun in different times. Regarding this as the limiting case of an outer planet seen from the Earth, we must make its orbit go just *outside* the Earth; if as the limiting case of an inner planet, it is just *inside*. Consider the former case: on leaving the Earth (E) the planet (P) makes the whole circuit of its orbit round the Sun, but the line EP only describes 180° until the Earth is nearly reached again, when the second 180° is described instantaneously in *the same* direction. Hence the mean motion of the planet, relatively to the Earth, is the same as its mean motion round the Sun. (We deal of course, with mean motion in space, *not* with synodic motion.) The revolution round the Earth is far from uniform: but it can be regarded as uniform motion with an impressed oscillation, the uniform motion being that of P.

If, however, as in the second case, we think of P as passing just *inside* E, then the instantaneous description of 180° is in the *opposite* direction to that which has just been laboriously made in travelling round the orbit, so that the work of this latter is completely undone. The planet P therefore on the average remains stationary within the orbit of E just as the Sun does, and the mean period of the relative motion is therefore that of the revolution of E round the Sun; on which, of course, is impressed an oscillation due to P.

16. The point to which attention is here drawn concerns the use of words rather than the interpretation of phenomena: but it is submitted with some seriousness that to speak of a periodicity of which the coefficient may regularly change sign is a misuse of words. The coefficient may die down, but it should keep the same sign: zero should be its limit: if it reverses in sign persistently (and not merely accidentally as an alternative to zero or a small positive value) then we should alter the adopted value of the period so as to avoid this reversal. In particular, protest is entered against the nomination of 88·7 days by Martin and Plummer as that of RS Camelopardalis: the adopted period should be that of the term with largest coefficient

$$\cdot 172 \sin (4^\circ \cdot 382t' - 25^\circ)$$

which is

$$360^\circ / 4^\circ \cdot 382 = 82 \cdot 2 \text{ days.}$$

Baxendell's Observations of Variable Stars. Edited by H. H. Turner and Mary A. Blagg. Eighth Instalment. No. 20, T Tauri; No. 21, R Ursæ Majoris; No. 22, S Ursæ Majoris.

No. 20, T Tauri (1537 Chandler).

$4^h\ 13^m\ 33^s\ +19^\circ\ 11'{\cdot}3\ (1855{\cdot}0)$
$4\ \ 16\ \ 10\ \ +19\ \ 17\ {\cdot}8\ (1900{\cdot}0)$

Identification of Comparison Stars.—Three charts and one small sketch have been found, in which the comparison stars have been identified as shown in Table I. In most cases the identification is fairly easy, but as none of the charts are dated, it can be decided only by internal evidence of the observations themselves which chart was in use at any particular time. There are a few cases of anomalous observations in which a slip seems to be suggested.

TABLE I.

Baxendell's Comparison Stars.

Baxendell.				A.S.V.		H.A.		Adopted Mag.
Large Charts. (1)	Large Charts. (2)	Small Chart.	Sketch, 1862 Nov. 15.	No.	Grade.	Letter.	Mag.	
a	*a*	*a*	*b*	2	13	*b*	8·27	...
b	*r*	*r*	...	3	33	*d*	9·30	93
c	*s*	*s*	...	7	43	*f*	10·10	101
d	*t*	*t*	...	9	48	...	...	105
e	*γ*	*γ*	...	10	51	*g*	10·76	108
f	*w*	*w*	...	11	55	...	...	109
g	*y*	*y*	*y*	...	...	...	...	110
h	*ζ*	...	...	...	...	...	...	116
k	*f*	...	...	...	...	...	...	125
l	...	...	*c*	13	76	*l*	12·37	124
m	Z	Z	*v*	16	93	...	...	128
n	...	...	*w*	...	...	...	...	130
p	*p*	*p*	...	8	44	...	...	102
...	...	*o*	...	...	...	...	...	122

TABLE II.

Baxendell's Differences between his Comparison Stars.

Years.	105–101.	109–108.	110–109.	116–109.	124–116.	128–124.	130–124.	102–101.	108–102.	109–102.	124–109.
1862–4	$1{\cdot}5_1$	$+0{\cdot}6_{14}$	$-0{\cdot}4_{15}$	$6{\cdot}3_3$	$12{\cdot}7_3$	$5{\cdot}0_2$	...	...	...	...	$19{\cdot}0_2$
1865–9	...	$+0{\cdot}3_{11}$	$+4{\cdot}2_3$	$10{\cdot}0_7$	$9{\cdot}1_6$	$4{\cdot}2_3$	...	...	...	...	$18{\cdot}8_2$
1877–81	$4{\cdot}5_2$	$-1{\cdot}5_1$	...	...	...	...	...	$1{\cdot}8_3$	$6{\cdot}2_3$	$7{\cdot}8_6$	...
1884–7	...	...	$+2{\cdot}0_1$	$14{\cdot}0_1$	$9{\cdot}5_2$	...	$7{\cdot}6_4$	...	...	...	...
Mean.	3·5	+0·4	+0·5	9·1	10·1	4·5	7·6	1·8	6·2	7·8	18·9
No. of Obsns.	3	26	19	11	11	5	4	3	3	6	4
Reduced Mean.	3·2	+0·4	+0·3	8·0	8·6	3·9	6·5	1·7	5·0	6·6	15·1

The unit is Baxendell's "step," reduced in the lowest line approximately to a tenth of a magnitude by the usual formula applied to each comparison. The telescopes used were A 68 during the earlier series, except on the following occasions:—1862 Nov. 15 and 16 (2nd observation) and 1864 Nov. 5; R (13) 81 was used: 1864 Oct. 6 and 7; R (13) 196: 1863 Aug. 19 and Sept. 19 and 1866 Sept. 15; C 39: 1863 Feb. 17, $7\frac{1}{2}$ in. A 88. In 1870 and 1871 Mr. Gladstone's $7\frac{1}{4}$ in. was used, with powers 60 and 69. From 1877, E 80 was almost certainly used, though this is not noted.

TABLE III.

Baxendell's Observations of T Tauri.

Date.	J.D.	Corrected Comparisons.	Deduced Mag.
1862	240+		
Mar. 19	1219	110−4	106
Oct. 18	1432	Invisible <125	
Nov. 15	1460	124−2, 128−7	122
16	1461	124−2 (2 obsns.)	122
Dec. 26	1501	124+0	124
1863			
Jan. 23	1529	124−5	119
29	1535	Invisible <102	
31	1537	Glimpsed, "☽ and v. hazy."	
Feb. 3	1540	Not seen <103	
8	1545	124−3	121
15	1552	124−4	120
16	1553	Glimpsed	
17	1554	124−4	120
21	1558	124−4	120
Mar. 3	1568	Invisible <105 "v. strong ☽."	
15	1580	124+0, 128−3	124
21	1586	124+0	124
22	1587	124−2	122
Aug. 19	1737	110+2, 109+2	112
24	1742	109+3, 116−3, 124−12	112
Sept. 19	1768	109+6	115
22	1771	109+6, 124−8, 116+2	116
Oct. 9	1788	109+6, 116+0	116
Nov. 2	1812	116+2, 124−8	117
29	1839	116+2	118
Dec. 17	1857	109−1, 108−1	108
1864			
Jan. 4	1875	108+0, 109+0	108
Feb. 5	1907	108+1, 109−1	108
8	1910	108+1, 109−1	108
Mar. 28	1959	108+1, 109+0	109
Sept. 24	2139	109+2, 108+0	110
Oct. 6	2151	109+3, 108+3	112
7	2152	109+2, 108+3	111
Nov. 5	2181	108+1, 109−1	108
Dec. 21	2227	108+1, 109+0	109
1865			
Feb. 14	2282	109−2, 108−2	106
Oct. 19	2529	108+1	109
Nov. 12	2553	108−1	107
1866			
Jan. 5	2607	108+11, 109+11	120
15	2617	124+3, 128−3	126
23	2625	Not seen <125	
Feb. 3	2636	124+2, 128−2	126
4	2637	124+3	127
12	2645	124+3	127
17	2650	124+3	127

TABLE III.—*Baxendell's Observations of T Tauri*—continued.

Date.	J.D. 240+	Corrected Comparisons.	Deduced Mag.
1866			
Mar. 13	2674	124+4,* 128+2	? 129
Sept. 15	2860	Not seen <110	
Oct. 6	2881	,, <125	
8	2883	Glimpsed.	
10	2885	124+2	126
16	2891	124+2	126
28	2903	About 124+0. "Low and seen with difficulty."	
30	2905	124+3	127
Nov. 11	2917	Not seen <120	
16	2922	124+3	127
30	2936	Not seen <127	
Dec. 7	2943	124+3	127
30	2966	124−3	121
31	2967	116+3, 124−5	119
1867			
Jan. 2	2969	116+2, 124−7	118
4	2971	116+1	115
8	2975	116+0, 124−8	116
11	2978	116+0, 124−8	116
21	2988	Not seen <105	
28	2995	109+3, 116−2	113
Feb. 2	3000	109+5, 116−3	114
20	3018	109+8, 116−2	116
28	3026	116−2, 124−7, 109+7	116
Mar. 16	3042	109+7, 116−2	115
20	3046	109+7, 116−2	115
26	3052	109+7, 116−3, 124−9	115
Nov. 17	3288	109+2, 108+0	110
Dec. 6	3307	108+4, 109+4	112
18	3319	108+3, 109+2	111
22	3323	108+3, 109+3	112
1868			
Jan. 26	3358	108−2, 109−2	106
Feb. 11	3374	108+2, 109+1	110
15	3378	109+0	109
1869			
Feb. 5	3734	109+0, 108+1	109
Mar. 2	3759	108+2, 109+3	111
24	3781	108+4, 109+3	112
Dec. 21	4053	Not seen <120	
1870			
Mar. 25	4147	124+6	130
1871			
Mar. 3	4490	Not sure if glimpsed. "☽ and hazy."	
Mar. 13	4500	124+6	130
1877			
Dec. 6	6960	102+0, 108−3	104
1878			
Jan. 31	7016	101+2, 105−3, 102−2, 108−5	102
Nov. 25	7314	102+2, 109+0	106
Dec. 5	7324	102+2, 109−5	104
23	7342	102+1	103
1879			
Sept. 13	7606	102+1, 109−5, 101+2, 105−2	103
Oct. 20	7643	102−1	101
Nov. 2	7656	102−2	100
1880			
Feb. 5	7751	93+5, 102−2, 101+1	99
Aug. 17	7945	102−3	99
Oct. 2	7991	102+0	102
1881			
Oct. 29	8383	109−2	107
Nov. 14	8399	? 109−2† (or 93−2)	? 107 (or 91)
Dec. 19	8434	102+6, 109−3	107

* "Has *m* changed ? or is it a star not before laid down that I have taken for *m* ?"
† The letter is *b*, but it seems probable that it is a slip for *f*.

TABLE III.—*Baxendell's Observations of T Tauri*—continued.

Date.	J.D.	Corrected Comparisons.	Deduced Mag.
1884	240+		
Jan. 20	9196	116+6, 124-7	120
Nov. 20	9501	125+0	125
Dec. 14	9525	About 124+2	126
15	9526	124+4, 130-2	128
20	9531	124+4, 130+0	129
1885			
Jan. 6	9548	124+3, 130-3	127
13	9555	124-3, 130-11	120
Feb. 2	9575	124-6, 122-4, 125-6	118
Mar. 17	9618	124-6, 125-7	118
Dec. 1	9877	116-2	114

Date.	J.D.	Corrected Comparisons.	Deduced Mag.
1886	240+		
Jan. 6	9913	109+6, 116-6	112
Feb. 1	9939	125-3	122
14	9952	About 124+0. "☽ too near."	?124
Mar. 3	9969	109+3, 110+1	112
	241+		
Nov. 2	0213	124+0. "v. low."	?124
30	0241	124-6, 116-3	116
Dec. 4	0245	116-3	113
29	0270	116-2	114
1887			
Jan. 29	0301	108+6, 116-4	113
Feb. 18	0321	116+0	116

The observations of 1886 appear to be inconsistent and puzzling. The variations of this star would appear to be irregular, but the observations are so scattered and the gaps so long and so frequent, that it is impossible to deduce even that conclusion with absolute certainty. Five minima seem to be indicated; one somewhere about 240 1500 to 1600 (or it might be earlier), one between 2670 and 2930, another ending about 9530, and two others, one near 4150 and the other near 4500. But in the first case there seems to be a rise of half a magnitude, lasting for about 40 days, between two separate lowest points if we suppose the actual minimum to be observed, and this might very well be the case in the other minima also. No maximum seems to be actually observed, but one probably occurred between 7800 and 7900.

No. 21, *R Ursæ Majoris* (3825 *Chandler*; 103769 *Harvard*).

$10^h\ 34^m\ 19^s\ +69^\circ\ 32'{\cdot}1\ (1855{\cdot}0)$
$10\ 37\ 34\ +69\ 18{\cdot}0\ (1900{\cdot}0)$

Identification of Comparison Stars.—There are four small sketches of the comparison stars, all in agreement with one another; and from 1863 Pogson's chart was used, some additional letters being inserted by Baxendell.

TABLE IV.

Comparison Stars.

Chart (Pogson's.)	Baxendell Sketches. 1857. Apr. 19.	1857. July 25.	1859. Jan. 6.	1862. Apr. 15.	A.S.V. No.	A.S.V. Grade.	Harvard. Mag.	Adopted Mag.	Bax.'s and Pogson's est.
a	*a*	*a*	...	*a*	2	...	5·90	63	6·7
c	*b*	*b*	*b*	*b*	5	14	8·25	81	8·1
d	*d*	*d*	...	...	7	31	9·19	93	9·8
e	...	...	...	...	12	52	10·28	101	...
f	...	...	...	...	10	43	9·83	100	10·4
e'	...	...	...	...	9	42	9·78	102	...
f'	...	...	...	...	...	...	...	106	...
g	...	...	...	...	13	65	10·89	110	11·2
h	...	...	...	...	14	72	11·20	114	11·5
k	...	...	...	...	18	91	11·94	119	12·5
l	...	...	...	...	23	114	12·61	126	12·6
...	*c*	*c*	...	...	...	...	...	...	...

The Harvard magnitudes are obtained by interpolation, and are very uncertain. The adopted magnitudes are a compromise.

A star called *m* is used once, in 1885; there is no clue to its identification.

TABLE V.

Baxendell's Differences between his Comparison Stars.

Years.	81–63.	93–81.	101–93.	100–101.	110–100.	114–110.	119–114.	126–119.	106–100.
1857–8	$12\cdot9_{21}$	$14\cdot7_5$	...	...	...	...	...	...	...
1861–7	$15\cdot5_2$	$20\cdot0_2$	...	$+3\cdot0_1$	...	...	..	...	$4\cdot0_1$
1879	...	$18\cdot0_1$	...	...	...	...	...	...	...
1884–6	$22\cdot4_7$	$16\cdot7_6$	$8\cdot8_2$	$-1\cdot2_2$	$11\cdot5_2$	$5\cdot5_2$	$4\cdot3_3$	$8\cdot8_2$	$12\cdot0_1$
Mean.	15·3	16·5	8·8	+0·2	11·5	5·5	4·3	8·8	8·0
No. of Obsns.	30	14	2	3	2	2	3	2	2
Reduced Mean.	12·0	13·5	7·2	+0·3	10·0	4·2	3·7	7·0	6·5

In this case there are so few comparisons for most of the stars that we give even those which occur only twice; but practically no weight attaches to any of the columns as regards the *exact* intervals, though something may be inferred as to the approximate relative differences.

The telescopes used were:—A (30) in 1857 Apr. to July, 1858 Mar. 11, Apr. 18, and June; A 40, 1858 Mar. 16 and 25; A (22) 21, 1858 Aug. to Dec., and 1859 (except Jan. 18); A 68, 1861 May 17 and June 25, 1865 June 18, and 1867 Mar. 16; Tully A 35, 1857 Dec., 1858 Feb. and Mar. 3; ditto 40, 1858 Jan. 25;

R (7) 40, 1858 Apr. 1 to 12 and July 18 and 1859 Jan. 18; R (13) 81, 1858 Mar. 23; C 39, 1861 Jan. 1, Feb. 28, May 10 and 30, June 17, Aug. 6, Sept. 1, 1862 to 1864, 1865 Nov. 12, and 1867 May 11; C 21, 1861 July 5. From 1879 E 80 was most probably used throughout.

TABLE VI.

Baxendell's Observations of R Ursæ Majoris.

Date.	J.D.	Corrected Comparisons.		Deduced Mag.
1857	239+			
Apr. 19	9424	81 – 1		80
20	9425	81 – 2		79
23	9428	81 – 2		79
25	9430	81 – 3		78
26	9431	81 – 3		78
27	9432	81 – 3		78
May 4	9439	81 – 5,	63+81 / 2	74
5	9440	63+9,	81 – 5	74
7	9442	63+7,	81 – 5	73
15	9450	63+5,	81 – 5	72
19	9454	63+4,	81 – 6	71
21	9456	63+5,	81 – 5	72
24	9459	63+7,	81 – 4	74
,,	,,	63+81 / 2		72
26	9461	63+6,	81 – 4	73
28	8463	63+5,	81 – 4	72
30	9465	*63+6½,	81 – 4	73
,,	,,	63+6,	*81 – 4½	73
June 4	9470	63+7,	81 – 4	74
12	9478	63+8,	81 – 3	74
14	9480	63+7,	81 – 3	74
15	9481	63+8,	81 – 3	74
17	9483	63+8,	81 – 2	75
19	9485	63+8,	81 – 3	74
20	9486	63+8,	81 – 2	75
21	9487	63+9,	81 – 2	76
22	9488	81 – 2		79

Date.	J.D.	Corrected Comparisons.		Deduced Mag.
1857	239+			
June 24	9490	63+9,	81 – 1	76
25	9491	63+9,	81 – 1	76
26	9492	63+9,	81 – 1	76
July 1	9497	81 – 1		80
11	9507	81+1		82
12	9508	81+2		83
19	9515	81+5		86
25	9521	81+7,	93 – 3	89
Dec. 7	9656	Glimpsed.		
20	9669	,,		
1858				
Jan. 25	9705	Invisible)		
Feb. 6	9717	93+1		94
7	9718	? †93+0		? 93
15	9726	93 – 3		90
22	9733	81+2		83
26	9737	81 – 1		80
Mar. 3	9742	81 – 2		79
11	9750	81 – 0		81
16	9755	81+0		81
20	9759	81 – 1		80
23	9762	81 – 1		80
25	9764	81+0		81
Apr. 1	9771	81+1		82
4	9774	81+1		82
12	9782	81+4,	93 – 7	86
18	9788	81+6		87
20	9790	81+7,	93 – 7	87
23	9793	81+6,	93 – 7	86

* The half-tenths are here given to show the difference between the two observations.

† Bax. wrote δ.

TABLE VI.—*Baxendell's Observations of R Ursæ Majoris*—continued.

Date.	J.D.	Corrected Comparisons.	Deduced Mag.
1858	239+		
June 6	9837	Invisible, "not v. clear."	
7	9838	,,	
8	9839	,,	
July 18	9879	101+3, min. vis. −4	104
Aug. 10	9902	Occas. glimpsed.	
Sept. 5	9928	Barely ,,	
11	9934	Invisible.	
Nov. 5	9989	,,	
	240+		
28	0012	93−3	90
Dec. 2	0016	93−5	88
1859			
Jan. 6	0051	81+0	81
18	0063	81−1	80
23	0068	81+1	82
28	0073	81+1	82
Feb. 6	0082	81+1	82
11	0087	81+1	82
23	0099	81+4	85
May 29	0194	Invisible.	
1861			
Jan. 1	0777	,,	
Feb. 28	0835	Est. 110?	
May 10	0906	81+11, 93−7	89
17	0913	81−2	79
30	0926	63+2, 81−9	68
June 17	0944	63+3	66
25	0952	63+2	65
July 5	0962	63+3	66
Aug. 6	0994	63+11, 81−2	76
Sept. 1	1020	81+6	87
1862			
Mar. 7	1207	Not seen <93	
Apr. 15	1246	81−5	76
27	1258	81−4	77
May 19	1280	81−2	79
1862	240+		
July 4	1326	Not seen <100 "v. hazy."	
Sept. 20	1404	Not seen <115	
1863			
Apr. 30	1626	,, <100	
Dec. 17	1857	81−3	78
1864			
Jan. 2	1873	81+0	81
1865			
June 18	2406	114−1	113
Nov. 12	2553	81+6, 93−8	86
1867			
Mar. 16	3042	101+6, 100+3, 106+0	105
May 11	3098	81−5	76
1879			
Aug. 28	7590	81+11, 93−3	91
Sept. 29	7622	81−3	78
1884			
May 26	9323	110+6	116
July 19	9377	100+4, 114−6	106
25	9383	93−2, 100−6, 101−9	92
Aug. 1	9390	81+4, 93−9	84
5	9394	81+0	81
11	9400	81−3	78
15	9404	81−6	75
23	9412	81−6	75
Sept. 1	9421	63+9, 81−8	72
4	9424	63+12, 81−8	74
11	9431	63+11, 81−7	74
13	9433	81−6	75
17	9437	81−4	77
20	9440	63+11, 81−6	74
26	9446	63+11, 81−7	74
29	9449	81−7	74
Oct. 1	9451	63+11, 81−7	74

TABLE VI.—*Baxendell's Observations of R Ursæ Majoris*—continued.

Date.	J.D.	Corrected Comparisons.	Deduced Mag.
1884	240+		
Oct. 4	9454	63+11, 81−7	74
5	9455	81−7	74
9	9459	81−6	75
12	9462	81−6	75
Nov. 2	9483	81+0	81
7	9488	81+1	82
9	9490	81+1	82
13	9494	81+6	87
17	9498	81+5	86
20	9501	81+6, 93−7	86
24	9505	93−7	86
27	9508	93−3	90
Dec. 3	9514	93+1	94
6	9517	93+3, 101−4, 100−6	96
20	9531	100+4, 110−5, 102+3, 106−6	104
25	9536	100+6, 110−4, 114−8	106
1885	240+		
Jan. 6	9548	110+0	110
12	9554	114−6, 110−1	109
13	9555	114−2	112
Feb. 19	9592	126+5	131
Apr. 2	9634	126+3, ?−3	?129
June 1	9694	81+8, 93−5	88
12	9705	81−5	76
July 4	9727	81−8	73
9	9732	81−6	75
25	9748	81−4	77
29	9752	81−2	79
Aug. 1	9755	81−2	79
28	9782	81+7, 93−7	87
Sept. 13	9798	93+1	94
1886			
Jan. 31	9938	126+2	128
	241+		
Apr. 26	0023	81−5	76

A period of 300 days was at first assumed, but it became clear that this was a little too short, so the observations were arranged in the usual manner with an assumed period of 302 days. The means for different groups of periods are as follows, the unit being 0·1 magnitude:—

	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.
A	89_6	79_{13}	$76\frac{1}{2}{}_{18}$	79_{19}	85_4	...	...	104_1	...	...	...	...
B	84_3	72_5	75_4	76_1	87_1	86_1	...	...	...	...	...	89_1
C	94_3	80_7	75_{10}	75_{10}	83_4	89_7	102_3	110_3	...	131_1	124_3	113_1
General Mean	89	78	76	78	84	89	102	108	...	131	124	10
No. of Obsns.	12	25	25	30	9	8	3	4	0	1	3	2

Group A contains Periods I. to IV.; group B Periods V. to XIII.; group C Periods XXVIII. to XXXVI.

There is some appearance of a retrograde movement of the maximum between A and B, and a forward movement between B and C, but the observations in B are too few to trust with any certainty. The observations in the second half of the curve are so few throughout that we cannot expect to obtain from them a very smooth curve, but the portion from 1 to 6 is fairly satisfactory. The only maxima which can be considered as being in any real

sense observed are those of Periods I., II., and XXXIV., and even of these the dates of the second and last cannot be fixed with any real certainty. The following table gives a series of epochs at fixed intervals of 301·9 days, compared with Baxendell, and with later observations of the B.A.A. Those of periods LVI. to LXIV. are as published, the *most* doubtful ones being omitted (though all are really doubtful to 8 or 10 days), and the later ones are calculated by the usual method from observations kindly lent to us in MS. by Mr. Brook. Comparisons with Chandler's formula $2397949 + 302{\cdot}1E + 11 \sin(8^\circ E + 238)$ are also given.

TABLE VII.

Comparison of Epochs of Maximum.

Period.	E.	Mean Epoch.	Observed ditto.	O – M.	Chandler.	O – Ch.
I.	5	9452	9454	+ 2	9449	+ 5
II.	6	9754	9745	– 9	9751	– 6
III.	7	0056	0060	+ 4	0054	+ 6
VI.	10	0962	0952	– 10	0963	– 11
XXXIV.	38	9415	9423	+ 8	9428	– 5
XXXV.	39	9717	9723	+ 6	9729	– 6
LVI.	60	6056	6059	+ 3	6075	– 16
LXI.	65	7566	7571	+ 5	7592	– 21
LXII.	66	7868	7879	+ 11	7896	– 17
LXIII.	67	8170	8177	+ 7	8199	– 22
LXIV.	68	8472	8475	+ 3	8502	– 27
LXV.	69	8774	8774	0	8804	– 30
LXVI.	70	9075	9078	+ 3	9107	– 29
LXVII.	71	9377	9373	– 4	9409	– 36
LXVIII.	72	9679	9676	– 3	9711	– 35
LXIX.	73	9981	9976	– 5	0013	– 37
LXX.	74	0283	0275	– 8	0315	– 40

The figures of the O – M column give no trustworthy evidence of any change of period either continuous or discontinuous, and Chandler's formula has clearly broken down. It could not be brought into agreement simply by reducing the length of the mean period while retaining the curve of variation.

No. 22, S Ursæ Majoris (4557 *Chandler*; 123961 *Harvard*).

$12^h\ 37^m\ 35^s\ +61^\circ\ 53{\cdot}3'\ (1855{\cdot}0)$

$12\ \ 39\ \ 34\ \ +61\ \ 38{\cdot}5\ \ (1900{\cdot}0)$

Identification of Comparison Stars.—In this case Pogson's chart was used from 1864 Feb. 17. Before that, four small

sketches and a chart are given, the earliest one containing a considerable number of letters, and the others very few. The identifications have been made as follows. The two queries are used in the sketches and in the observations as if they were letters.

TABLE VIII.

Baxendell's Comparison Stars.

1857 May 24.	1859 Feb. 6.	1859 May 29.	1861 Apr. 9.	Provisional Chart. 1863 Nov. 1.	Pogson's Chart.	A.S.V. III. No.	A.S.V. III. Grade.	A.S.V. IV. No.	A.S.V. IV. Grades.	Harvard. Letter.	Harvard. Mag.	Adopted Mag.
a	...	...	...	...	*a*	1	0	2	20, 11	*a*	6·65	...
e	...	...	...	...	*b*	2	13	3	31	*b*	7·02	73
f	...	...	...	...	*c*	5	29	4	37	*c*	7·30	74
(?) *x*	...	...	...	...	*d*	3	24	7	40, 43	*d*	7·52	77
g	...	...	...	*g*	*e*	6	51	11	70, 81	*f*	8·50	85
h	*h*	...	...	*h*	*f*	8	66	13	78, 92	*g*	8·83	88
...	...	...	...	...	*g*	14	122	23	117, 121	...	...	100
...	...	...	...	...	*h*	19	162	27	127, 129	...	...	105
l	...	...	*l*	*x*	*k*	16	139	24	122, 129	*m*	10·65	106
m	...	?	...	...	*l*	22	182	29	131, 133	*n*	11·08	108
...	...	...	...	...	*m*	26	217	30	143, 140	*o*	11·58	116
c	...	...	...	...	...	...	...	9	64, 66	...	...	...
k	...	...	?	...	..	13	118	21		*l*	10·09	101
...	...	...	...	...	(*n*)	28	230	31	148, 145	*p*	11·94	120

The letter *n* is used only once, with nothing to identify it, but it seems highly probable that it is A.S.V. (IV.) No. 31, as it is used in conjunction with the neighbouring star, No. 30. The magnitudes adopted are those of Harvard (xxxvii. p. 7), with slight modifications.

TABLE IX.

Baxendell's Differences between his Comparison Stars.

Years.	77–74.	85–77.	85–74.	88–85.	101–88.	106–101.	108–106.	108–101.	116–108.	100–88.	105–100.
1857–8	...	? $5{\cdot}0_1$	$3{\cdot}0_1$	$2{\cdot}3_6$	$7{\cdot}3_2$	$4{\cdot}2_5$	$1{\cdot}8_3$	$6{\cdot}5_3$	...	...	...
1861–7	...	?$17{\cdot}0_1$	$14{\cdot}0_3$	$3{\cdot}6_7$	$9{\cdot}0_1$	$4{\cdot}7_3$	...	...	...	...	...
1879–85	$3{\cdot}3_7$	$9{\cdot}8_{17}$	$12{\cdot}3_5$	$3{\cdot}8_7$	...	...	$3{\cdot}0_1$	...	$10{\cdot}1_4$	$18{\cdot}0_6$	$6{\cdot}3_3$
Mean.	3·3	9·9	11·8	3·3	7·8	4·4	2·1	6·5	10·1	18·0	6·3
No. of Obsns.	7	19	9	20	3	8	4	3	4	6	3
Reduced Mean.	3·0	8·3	10·0	2·9	6·8	3·8	1·9	5·5	8·8	14·2	5·5

In most cases the number of comparisons is too small to give trustworthy results.

The telescopes used were:—Tully 35: 1857 Dec. 20; 1858

Feb. 22, 24 (2nd obs.), 26, Mar. 3; A (30): 1857 Dec. 7, 1858 June 7 to 15, June 20, 26, 27, Aug. 5 and 8; A 68: 1858 Mar. 11, 1861 Mar. 21, Apr. 6, May 17, June 25, 1863 Nov. 1, 1864 from Jan. 4, 1866 June 26, and Dec. 16 to 1867 Mar. 16; A 40: 1858 Mar. 16 to 25; R (13): 1858 Feb. 24 (1st obs.); R (7) 40: 1858 Apr. 1 to June 4, June 17, 21, July 18; A (22) 21: 1858 July 1 to 17, 19, Aug. 10 to 1859 May 29; C 39: 1861 Jan. 1 to Mar. 8, Mar. 29, 31, Apr. 9 to May 10, May 30, June 17, Aug. 6 to 1863 Apr. 30, 1866 Aug. 22; E 80 (probably) from 1879.

The variable was first ostensibly observed 1857 April 19, and further observations followed on six dates, ending with May 15. On the first two occasions the comparisons are with 76 Urs. and 75 Urs.; on the next two, with these and also with Gr. 1918 (later called *a*); and on the last three with the latter only. The resulting magnitudes for the supposed variable range from 6·2 to 6·3; several of the observations are marked with a query; they are all written in pencil only (all others of the same dates being in ink); and, on the last of these dates, a note is added "? is this the real S Ursæ." It seems clear that it could not have been so. On May 24 the sketch is given, and comparisons are made between certain of the comparison stars on that date, on May 26 and 30, and on June 4. These comparisons are made with A (30), and the intervals come out very small compared with those obtained with the instruments in use later on. The real S Ursæ is first observed 1857 Dec. 7.

TABLE X.

Baxendell's Observations of S Ursæ.

Date.	J.D.	Corrected Comparisons.		Deduced Mag.	Date.	J.D.	Corrected Comparisons.		Deduced Mag.
1857	239+				1858	239+			
Dec. 7	9656	85+0,	?77+4	?83	Apr. 20	9790	Not seen, moonlight.		
20	9669	85+0		85	23	9793	,,	,,	
1858					May 21	9821	106+1		107
Feb. 22	9733	85+4		89	June 4	9835	88+3,	101−4	94
24	9735	85+3		88	7	9838	88+3		91
,,	,,	85+4,	88+3	90	8	9839	88+3		91
26	9737	88+3		91	12	9843	88+2		90
Mar. 3	9742	88+6,	101+1	98	15	9846	88+0		88
11	9750	101+2,	106−1	104	17	9848	88+1		89
16	9755	101+2,	106−2	104	20	9851	85+1		86
20	9759	106+0		106	21	9852	85+0		85
25	9764	106+1		107	26	9857	85+0		85
Apr. 1	9771	108+0		108	27	9858	85+0		85
4	9774	108+0		108	July 1	9862	85−1		84
9	9779	108+3		111	6	9867	85−1		84
12	9782	108+4		112	10	9871	85−2		83

TABLE X.—*Baxendell's Observations of S Ursæ*—continued.

Date.	J.D.	Corrected Comparisons.	Deduced Mag.
1858	239+		
July 17	9878	85−3	82
18	9879	85−3	82
19	9880	85−2	83
Aug. 5	9897	85−2	83
8	9900	85−1	84
10	9902	85−1	84
12	9904	85+0	85
Sept. 5	9928	85−1	84
11	9934	85−1	84
1859	240+		
Jan. 6	0051	88+7	95
Feb. 6	0082	88+0	88
11	0087	88+0	88
Mar. 31	0135	85+0	85
May 29	0194	108+0	108
1861			
Jan. 1	0777	85+2	87
3	0779	85+2, 88−2	86
29	0805	85−3	82
Feb. 6	0813	85−3	82
12	0819	85−2	83
24	0831	85−2, 88−4	84
28	0835	85+2, 88−1	87
Mar. 8	0843	88+0	88
21	0856	88+3	91
29	0864	88+9	97
31	0866	88+11, 106−2	102
Apr. 6	0872	88+12, 106−2	102
9	0875	106−3, 101+0	102
16	0882	101+3, 106+0	105
May 10	0906	Invisible <106+3	
17	0913	106+6	112
30	0926	Invisible <106+3	
June 17	0944	,, <88, "hazy, D, and twilight."	
25	0952	88+6, 101−2, 106−8	97

Date.	J.D.	Corrected Comparisons.	Deduced Mag.
1861	240+		
Aug. 6	0994	85+1, 88−3	86
Sept. 1	1020	?77+8, 85−6	?82
Oct. 5	1054	85−1	84
16	1065	85+3	88
Nov. 7	1087	88+4	92
1862			
Mar. 7	1207	88+0	88
Apr. 15	1246	85−6	79
27	1258	85−4	81
May 19	1280	85−1	84
July 4	1326	Not seen <100	
Sept. 20	1404	88+7	95
1863			
Apr. 30	1626	Not seen <100	
Nov. 1	1811	106+11	117
Dec. 2	1842	88+11	99
17	1857	88+3	91
1864			
Jan. 2	1873	88+0	88
4	1875	85+1, 88−1	86
Feb. 17	1919	74+7, 85−6	80
Mar. 15	1946	74+7, 85−4	81
July 8	2061	106+6	112
1865			
Nov. 12	2553	85+3, 88−3	86
1866			
June 26	2779	85−2	83
Aug. 22	2836	85−2	83
Dec. 16	2952	88+6, ?106*+0	?100
1867			
Jan. 28	2995	85+1, 88−1	86
Mar. 16	3042	74+7, 85−4	81
1879			
Sept. 23	7616	77+5, 85−4	82
28	7621	77+4, 85−4	81

* The query is Baxendell's.

TABLE X.—*Baxendell's Observations of S Ursæ*—continued.

Date.	J.D.	Corrected Comparisons.	Deduced Mag.
1879	240+		
Oct. 15	7638	88+1	89
Nov. 1	7655	88+10, 100−2	98
23	7677	106+0	106
Dec. 24	7708	116+0	116
1884			
May 27	9324	106+14	120
Aug. 6	9395	85−2	83
11	9400	85−1, 88−3	84
15	9404	85−3	82
23	9412	88−2	86
Sept. 1	9421	77+6, 85−3	82
4	9424	77+4, 85−4	81
11	9431	77+1, 85−9	77
12	9432	77+1, 85−6, 88−8	79
17	9437	85−8, 77−3	76
20	9440	77+0, 85−8, 88−9	78
26	9446	77−2, 85−5, 74+5	78
Oct. 1	9451	77+1, 85−8, 74+1	77
5	9455	77+3, 74+4, 85−9	78
9	9459	77+3, 85−5, 74+4	79
12	9462	77+3, 85−5, 74+6	80
Nov. 2	9483	88+4, 100−8	92
7	9488	88+11, 100−5	97
Nov. 9	9490	100−2	98
13	9494	100−2	98
17	9498	100−4	96
20	9501	100−5	95
24	9505	100−2	98
27	9508	100−2	98
Dec. 3	9514	100−1, 105−5	100
6	9517	100+4, 105−3	103
14	9525	106+2, *108+0	?108
20	9531	106+5, 108+3	111
1885			
Jan. 5	9547	108+1, 116−6	110
11	9553	108+4, 116−3	112
Feb. 19	9592	88+7, 100−10	92
22	9595	88+5	93
Mar. 27	9628	77+4, 85−5	80
Apr. 2	9634	77+2, 85−5	80
May 11	9673	73+2, 74+1, 77−3	75
June 9	9702	88+0, 100−11	88
July 9	9732	100+0	100
25	9748	106+3, 108+1	109
Aug. 1	9755	108+7, 116−6	112
20	9774	116+2, 120−4	117
Sept. 13	9798	108+6, 116−2	114
27	9812	105+1, 106−1	106
Oct. 29	9844	88+7, 100−9	93
Nov. 28	9874	77+5, 85−3	82
Dec. 26	9902	77+3, 85−4	80

There are, again, very long gaps in this series, but a period of $226\frac{1}{2}$ days was assumed, and the observations were arranged and tabulated in the usual way. The following are the results for different groups of periods:—

* The letter is *h*, but it seems probable it should be *l*, and this correction has been made accordingly.

	(1.)	(2.)	(3.)	(4.)	(5.)	(6.)	(7.)	(8.)	(9.)	(10.)	(11.)	(12.)
A	85	82	83	84	86	92	103	(107)	112	...	98½	89
B	(81)	(80)	82	(81)	(89)	(98)	(106)	(117)	(116)	105½	89	86
C	(93)	85	82	(78)	78	84	(96)	98	108	115	(114)	(105½)
Weighted Mean with C shifted.	83	81	81	83	89	96	106	114	114	105½	94	87

Group A contains Periods II. to IX., group B Periods XI. to XVII. and XXXVII., group C Periods XLIV. to XLVII.

The brackets show that one period only is observed in that portion. The maximum is in approximately the same position in the first two groups, but in the third group it has advanced about two portions.

The mean has therefore been obtained from these figures with those of the C group shifted back two portions. The result is a good, smooth curve, nearly equal in the rise and fall, but with the maximum much flatter than the minimum.

The individual observations have been compared with the curve, omitting those which are very scattered, and Table XI. shows the results for each portion of the different periods. The shift of 38 days has been applied to the observations in the C group, before comparing them with the curve.

TABLE XI.

Comparison of the Observations with a Mean Light Curve.

Period.	Initial Date.	Reference Numbers and Mean Magnitudes of Portions. (1.) 83.	(2.) 81.	(3.) 81.	(4.) 83.	(5.) 89.	(6.) 96.	(7.) 106.	(8.) 114.	(9.) 114.	(10.) 106.	(11.) 94.	(12.) 87.
II.	9626·5	...	[+2]	[+4]	...	...	−9	−4	−6	[−4]	...	+11	+2
III.	9853	+1	[+1]	[+3]	...	−2	...	...	...	...	...	+1	...
IV.	0079·5	+4	...	+3	...	...	...	+6	...	...	...	...	...
VII.	0759	+5	[+5]	[+1]	0	0	0	−1	...	[−4]	...	+1	...
VIII.	0985·5	+2	[+1]	...	0	+1	−3	...	...	...	...	...	+2
IX.	1212	...	[−2]	[0]	+1	...	...	...	...	...	...	−2	...
XI.	1665	...	...	...	...	...	...	...	+2	...	−8	−6	−2
XII.	1891·5	...	[−1]	[−1]	...	...	...	...	...	...	+2	...	...
XXXVII.	7554	...	...	...	−2	0	+3	0	...	[0]	...	...	...
XLIV.	9139·5	...	...	...	...	...	...	...	[+5]	...	...	...	−2
XLV.	9366	+1	[−3]	[−3]	−2	+7	+2	+2	[−4]	...	...	−6	−5
XLVI.	9592·5	−5	...	[−6]	+4	...	+5	+4	+2	+3	+2	...	+5
XLVII.	9819	0	...	[0]	...	...	...	...	...	...	...	...	...
$p-f$		+6	+2	−2	−8	−13	−17	−18	−8	+8	+20	+19	+11

Very few periods are sufficiently observed to give anything like trustworthy results when the observations are dealt with in the usual manner for corrections to epoch. This has been done for Periods IV., VII., VIII., IX., XLV., and XLVI., taking the observations so as to have the date of maximum near the middle

in each case. The observations in brackets have not been used, being too close to maximum or minimum.

These results being so scanty, it seemed well to compare them with those of later observations, and use has been made of the B.A.A. results published in 1913 and of later ones not yet published which have been kindly lent to us in MS. by Mr. C. L. Brook. The Baxendell dates in Table XII. are got by calculation, by the method used throughout this series, and so are the last six of the B.A.A. The others are as estimated in the B.A.A. Memoirs, such epochs as seemed very uncertain being omitted. It is quite evident that no continuous and unchanging length of period will suit all the dates, and this is still true when all reasonable allowance is made for possible errors of observation and flatness of maxima. A solution by the discontinuity method * suits very well, and so does one by a new formula

$$2400589{\cdot}4 + 226{\cdot}2\mathrm{E} + 28\sin(3^{\circ}{\cdot}6\mathrm{E} - 43^{\circ}).$$

Chandler's formula for this star was

$$2400571 + 226{\cdot}5\mathrm{E} + 35\sin(5^{\circ}{\cdot}4\mathrm{E} + 194^{\circ}).$$

TABLE XII.

Comparison of Epochs of Maximum.

E.	Observed Epoch. (Bax. or B.A A.)	Epoch by Disc.	Epoch by New Formula.	Chandler's Epoch.	O−D.	O−F.	O−Ch.
	240+						
−2	0115	0116	0116	0116	−1	−1	−1
+1	0799	0798·5	0798	0786	+0·5	+1	+13
2	? 1027	1026	1026	1009	+1	+1	(+18)
3	1254	1253·5	1253	1233	+0·5	+1	+21
39	9437	9436	9439	9429	+1	−2	+8
40	9664	9661	9665	9658	+3	−1	+6
	241+						
66	5516	5511	5511	5514	+5	+5	+2
71	6636	6636	6634	6631	0	+2	+5
74	7310	7311	7309	7304	−1	+1	+6
75	7540	7536	7534	7529	+4	+6	+11
76	7756	7761	7759	7753	−5	−3	+3
77	7988	7986	7984	7979	+2	+4	+9
79	8430	8436	8434	8430	−6	−4	0
81	8880	8886	8885	8883	−6	−5	−3
82	9108	9111	9111	9109	−3	−3	−1
83	9339	9336	9337	9336	+3	+2	+3
84	9556	9561	9562	9564	−5	−6	−8
85	9786	9786	9788	9791	0	−2	−5
86	? 0011	0011	0014	0019	+0	−3	−8

* Periods IV. to XLII. (E=36) at 227·5 days; the rest at 225 days.

Some of the residuals in the last column are large, and several of them can hardly be reconciled with the observations, but one is put in brackets because the "observed" epoch in that case is extremely doubtful, and Chandler's date is not impossible in face of the observations.

Analysis of the mean light curve gives the following expression:—

$$1{\cdot}666\cos(\theta - 223^\circ) + 0{\cdot}347\cos(2\theta - 90^\circ) + 0{\cdot}085\cos(3\theta - 349^\circ),$$

which becomes, when divided by the range and multiplied by 6,

$$2{\cdot}87\cos(\theta - 223^\circ) + 0{\cdot}59\cos(2\theta - 90^\circ) + 0{\cdot}15\cos(3\theta - 349^\circ).$$

The separate groups of periods are not sufficiently well observed to be worth analysing.

Analysis of the B.A.A. mean light curve, from the observations in MS., gives the expression

$$1{\cdot}654\cos(\phi - 149^\circ) + 0{\cdot}383\cos(2\phi - 294^\circ) + 0{\cdot}118\cos(3\phi - 172^\circ),$$

which becomes, when divided by the range and multiplied by 6,

$$2{\cdot}92\cos(\phi - 149^\circ) + 0{\cdot}68\cos(2\phi - 294^\circ) + 0{\cdot}21\cos(3\phi - 172^\circ).$$

If now we write $\phi = \theta - 74^\circ$, the expression becomes—

$$2{\cdot}92\cos(\theta - 223^\circ) + 0{\cdot}68\cos(2\theta - 82^\circ) + 0{\cdot}21\cos(3\theta - 394^\circ).$$

This is in good agreement with that of Baxendell, and the difference in phase agrees pretty well with the corrections to epoch. The magnitude at maximum and minimum is almost the same in the two curves, but the range is about 0·1 magnitude less in that of the B.A.A.

Daylight Photography of Stars as a means of testing the Equivalence Postulate in the Theory of Relativity. By A. F. and F. A. Lindemann.

One of the most important questions in physics is whether there is any interaction between gravitation and electromagnetic phenomena, *e.g.* whether light is affected by a gravitational field.

The principle of relativity leads to the conclusion that the energy E possesses inertia equal to $\frac{E}{C^2}$ if C is the velocity of light. Experiments with a pendulum consisting in large part of uranium, and which therefore contained energy at least of the order $2 \times (10)^{17}$ $\frac{\text{ergs}}{\text{gramme}}$, showed no deviation from the rule that weight is proportional to inertia, though the inertia is obviously increased by some 0·02 per cent. It seemed to follow that energy must be acted upon by gravity, and it was in order to treat this question

quantitatively that Einstein formulated his equivalance postulate, namely, that there is no physical means of distinguishing between a uniform gravitational field and a field of uniform acceleration. This postulate may be shown to imply that a ray of light passing a mass M at distance r will be bent out of its course by the angle $\alpha = \frac{4kM}{c^2 r}$, k being the gravitational constant.* As this angle is $\frac{0''.46}{\beta}$ in the case of a star $\beta°$ from the sun, experimental verification seems to be within the bounds of possibility.

Einstein himself suggested the possibility of determining the apparent position of fixed stars near the sun during a total eclipse. The rarity of total eclipses, however, and the large number of other problems, which can only be investigated during the few minutes of totality, seemed to render it desirable to try to work out a method of more universal applicability.

Two possibilities suggest themselves: observation and photography. Though it will be seen that the observation of stars near the sun is not impossible in certain cases, if suitable precautions are taken, the photographic method seems far superior. If it is possible to photograph stars near the sun, there are various ways of determining whether there is any apparent shift or not. Firstly, it might be possible to photograph the sun on the same plate as the star, and determine the distance from the limb. Apart from the difficulty of making an accurate measurement, this method requires elaborate apparatus for keeping the sunlight off the plate for most of the time and exposing the plate to the sun for a very short time. Another method would be to photograph a number of stars on the same plate and determine their relative positions. This relies upon there being a cluster of sufficiently bright stars near the ecliptic. A third possibility would be to photograph the star near the sun in daytime and photograph some other stars, whose positions are accurately known, on the same plate at a given time at night. A fourth method, which has many advantages, would be to photograph the same star every day as it approaches and recedes from the sun with the same setting of the telescope, shifting the plate, say, 0·5 cm. each time. If there is no interaction between gravity and light, the images should lie upon a regular curve approximating to a straight line. If Einstein's theory is correct, however, the curve will not be regular when the light from the star passes close to the sun.

The instrument at our disposal, a six-inch equatorial of 223 cm. focus, is of course not sufficient to carry out the experiments, as 1″ only corresponds to $\frac{1}{90}$th of a millimetre. It is to be hoped, however, that the question will be investigated by an observatory more fortunately situated, since, as the following will show, daylight photography of stars is by no means difficult.

* A. Einstein, "Grundlage der allgemeinen Relativitätstheorie." In his first paper on the subject Professor Einstein gave half these values.

Whether faint stars can be photographed or not is largely a question of the steadiness of the air. According to Pickering, a 0th magnitude star gives about as much light as a candle at 526 metres, *i.e.* it is equivalent to about $3{\cdot}6\ (10)^{-6}$ candle-power.

The diffraction disc of a star is $2{\cdot}4\ \frac{\lambda L}{D}$ in diameter, λ being the wave-length, L the focal length, and D the diameter of the objective. If the light from the star were concentrated upon a disc of this size, the surface brightness of a first-magnitude star would be

$$\frac{D^4}{5{\cdot}76\lambda^2 L^2} \cdot 3{\cdot}6 \cdot 10^{-6} = 2{\cdot}5 \cdot 10^2 \frac{D^4}{L^2}$$

if λ is equal to $5\ (10)^{-5}$ cm. This is of the order $8{\cdot}8\ (10)^4$ for a 150 cm. mirror of 1200 cm. focal length. An ordinary photographic plate of comparatively high sensitiveness requires 0·033 $\frac{\text{candle-power second}}{\text{cm.}}$ to render it developable. Thus a 16th magnitude star should be photographed in one second with the 60″ mirror at Mount Wilson, or a 27th magnitude star in seven hours. In reality the limit is about the 22nd magnitude.

It is difficult to give any very satisfactory explanation of this discrepancy. It seems that in practice the smallest stars taken into account are those whose image is about $\frac{1}{10}$th of a millimetre. It may be that this is the limit of instrumental accuracy, or it may be that the grain of the plates is too coarse to permit accurate work with smaller discs; again, it may be, and in certain cases certainly is, that the atmosphere is too disturbed to allow a smaller image than one $(10)^{-2}$ cm. in diameter to be formed. If one calls d the diameter of the disc on the plate, the surface brightness of a star of nth magnitude would be

$$\frac{D^2}{d^2} \cdot 2{\cdot}512^{-n} \cdot 3{\cdot}6 \cdot 10^{-6}.$$

An exposure of t seconds would give rise to a developable image of diameter d if

$$t \cdot \frac{D^2}{d^2} = 9{\cdot}3 \cdot 10^3 \cdot 2{\cdot}512^n.$$

In other words, a star of nth magnitude will be photographed in t seconds if

$$n = 2{\cdot}5 \log t + 5 \log D - 5 \log d - 9{\cdot}9$$

if plates are used that are sensitive to $\frac{1}{30}$th of a candle-power second. It will be seen that, putting $d = (10)^{-2}$, D = 150 cm. and $t = 4\ (10)^4$ seconds, $n = 22$, in agreement with results at Mount Wilson.

In order to photograph stars in the daytime, the light from the stars must be comparable to the light from an equivalent solid angle of sky. The blackening of a plate may be expressed as a

function of the light × time by a curve of the type shown in fig. 1. The most effective exposure is obviously that at which the chemical action of the sky light is about at the point marked A, *i.e.* at which the plate is just about to become fogged. At this point the small extra amount of light from the stars makes most difference.

The scattered light from the sky may be represented with considerable accuracy by Rayleigh's formula as long as the particles

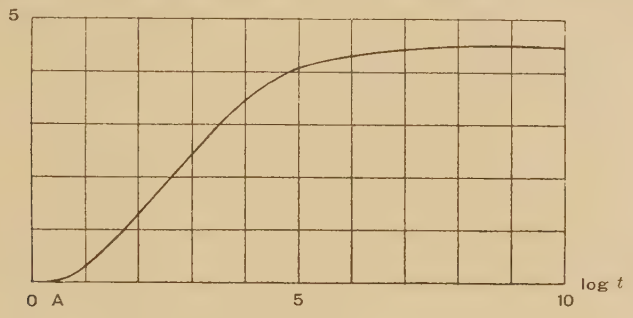

FIG. 1.

are small in comparison to the wave-length of light, *i.e.* as long as the sky is free from dust or clouds. In a blue sky the light from unit solid angle of sky at an angle β from the sun and γ above the horizon is given as *

$$i = \pi^2 \mathrm{I} \frac{m}{\mathrm{N}} \cdot \frac{p}{g} \cdot \frac{1+\cos^2\beta}{\cos\gamma} \left(\frac{\mu^2-1}{\rho}\right)^2 \frac{1}{\lambda^4} = \frac{1{\cdot}01 \cdot 10^{-19}}{\lambda^4} \cdot \frac{1+\cos^2\beta}{\cos\gamma} \mathrm{I},$$

I being the illumination due to the sun, m the mean molecular weight of the air, N the number of atoms in a gramme atom, p the pressure, g the gravitational acceleration, μ the refractive index, and ρ the density of air and λ the wave-length. It will be seen from this that in a perfectly clear sky the light from a star of magnitude n is equal to the light from an equivalent area of sky when

$$\frac{\pi^2}{16} \cdot \frac{d^2 \mathrm{D}^2}{\mathrm{L}^2} \cdot \frac{1{\cdot}01 \cdot 10^{-19}}{\lambda^4} \cdot \frac{1+\cos^2\beta}{\cos\gamma} \mathrm{I}$$

is of the same order as

$$\frac{\mathrm{D}^2}{d^2} \cdot 2{\cdot}512^{-(26{\cdot}6+n)} \mathrm{I},$$

assuming that the sun is of stellar magnitude −26·6, and further, that the energy distribution in the star's spectrum is something

* Strictly speaking, this quantity should be increased by the fraction of the total incident light reflected by the surface of the earth. This correction, which may of course be quite appreciable, has been neglected, as the formula is used only to give a rough idea of what is to be expected.

like that in the solar spectrum. In other words, the part of the photographic plate on which the image of the star falls will receive twice as much light as the surrounding parts if

$$n = 5 \log L + 10 \log \lambda - 10 \log d - 2{\cdot}5 \log \frac{1+\cos^2\beta}{\cos\gamma} + 21{\cdot}4.$$

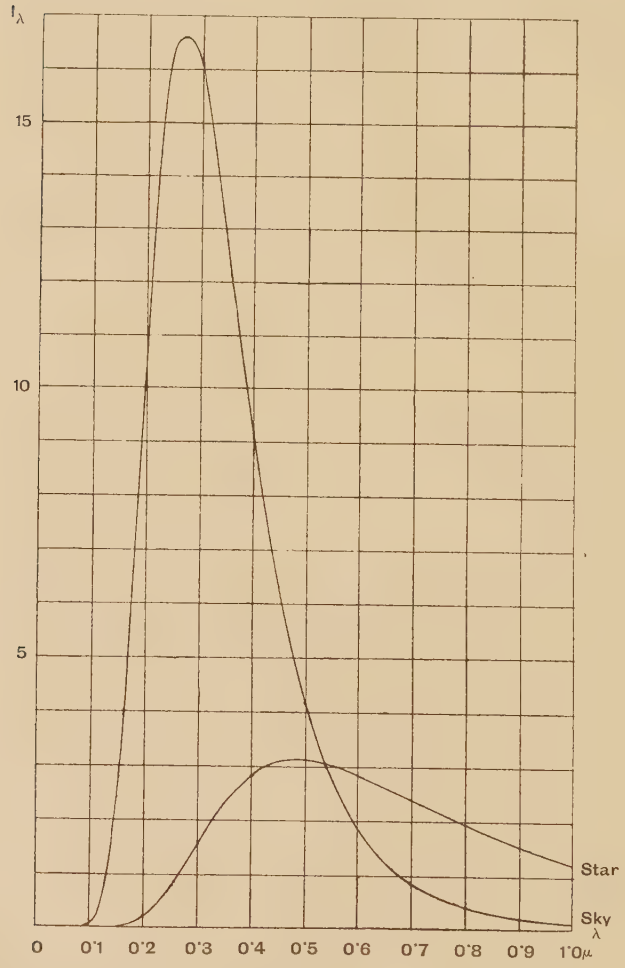

FIG. 2.

If the size of the image d is fixed, it is obviously of advantage to use as large a focal length as possible. If the size of d is determined by the scintillation of the star, $\frac{d}{L} = \delta$ will be independent of

the instrument and will represent the apparent angular displacement of the star. In that case

$$n = 10 \log \lambda - 5 \log L - 10 \log \delta - 2{\cdot}5 \log \frac{1 + \cos^2 \beta}{\cos \gamma} + 21{\cdot}4,$$

so that a short focus objective is to be recommended. The one change which always improves the contrast is to increase the wave-length λ. Thus a clear gain of 1·5 magnitude is effected by increasing the wave-length some 40 per cent. This becomes evident when the diagram (fig. 2) is considered. The curves are arranged to show the distribution of radiant energy in the spectrum of the sky and of a star of solar type, the scale of the ordinates being fixed so that the total light between 4000 Å. and 8000 Å., *i.e.* the total visible light from either source, is equal. It is clear how much is to be gained by using the red end of the spectrum.

The experiments were carried out with the above-mentioned equatorial telescope in the spring and summer of 1916. The apparatus consisted of a dark slide arranged so that different parts of the plate could be exposed in succession. A slide containing a number of different filters was mounted in front of the plate, so that various filters could be interposed and their relative usefulness compared. The plates used were Wratten Panchromatic B, and the filters were supplied by the same firm.* Those found most advantageous were those numbered 70 and 91. Filter 88 demanded rather long exposures. The percentages of light of different wave-lengths transmitted, as measured by Messrs. Wratten & Wainwright, are given in Table I.

Table I.

Filter No.	70. per cent.	91. per cent.	88. per cent.
6000			
6100			
6200			
6300			
6400			
6500	0·4		
6600	7·6		
6700	19·6	0·4	
6800	34·0	2·8	
6900	39·0	12·0	0·9
7000	45·5	15·5	6·5
7500	about 70	about 70	about 70

* We are indebted to the Wratten Division of Messrs. Kodak for the trouble they took in obtaining information about the constants of these plates and filters.

Fig. 3 indicates, on a scale five times the scale of fig. 2, the relative energy of the various wave-lengths in the light from a star and from the sky after passing through filter 70. Fig. 4 represents their effect in arbitrary units upon a Wratten Panchromatic B plate, assuming the sensitiveness to different wave-lengths

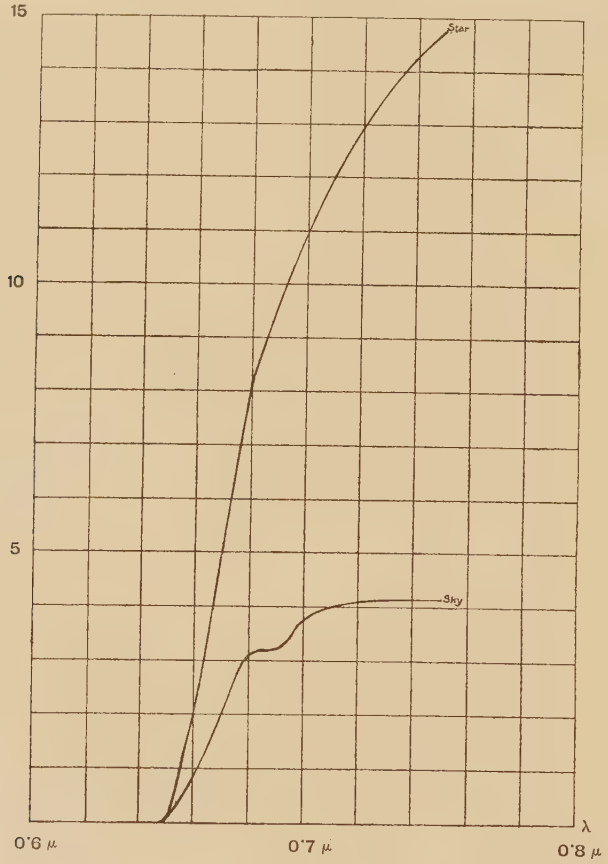

Fig. 3.

claimed by the makers. Fig. 5 shows the effect on an ordinary plate without a filter, using the figures given by Eder. It is apparent that whereas in fig. 4 the effect on the plate due to the star is about 2·75 times as great as that due to the light from the sky, it is about 2·75 times weaker in fig. 5, *i.e.* over two magnitudes have been gained by using a red sensitive plate and filter.

In practice it was found that the theoretical predictions were borne out with fair accuracy. Thus it was possible to photograph

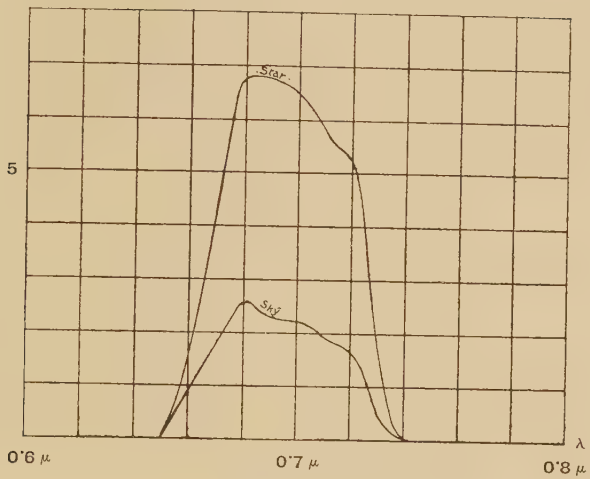

FIG. 4.

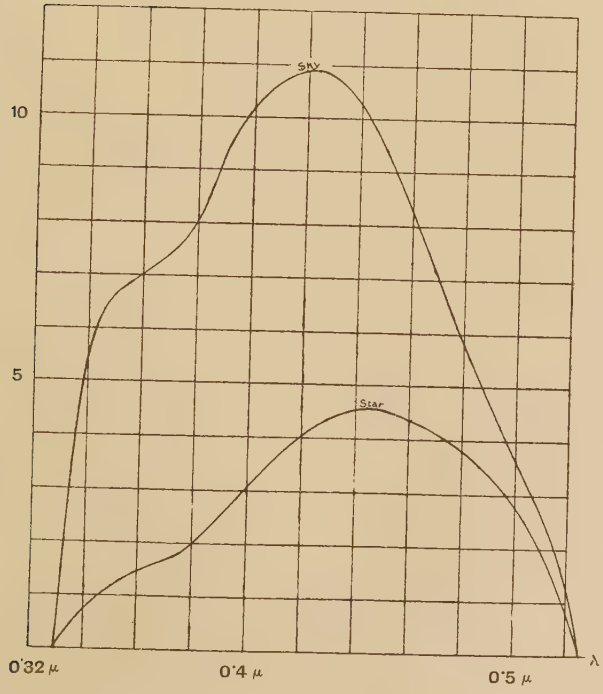

FIG. 5.

oth magnitude stars at any time of the day, even in hazy weather, whereas second and third magnitude stars required a very clear sky and calm air. The accompanying Table II. contains lists of a number of stars photographed, with details as to time of day, exposure, etc., which may be of use to anybody who may carry on this research.

TABLE II.

Date.	G.M.T.	Star.	Magnitude.	Filter.	Exposure (in secs.).	Sky.	Image.	No. of Plate.
7/6/16	5.55 p.m.	Arcturus	0·2	70	12, 16	Medium	Clear	15
18/6/16	11.19 ,,	,,	0·2	88	720, 480	,,	,,	17
19/6/16	6.49 ,,	,,	0·2	88	540	,,	Faint	20
30/7/16	12.15 ,,	Capella	0·2	70	5, 10, 15	,,	Clear	25
31/7/16	11.0 a.m.	,,	0·2	70	1,2,3,4,5	,,	4 & 5 clear	26
1/8/16	9.15 p.m.	Arcturus	0·2	70	4, 5, 6	Good	All ,,	29
1/8/16	9.20 ,,	,,	0·2	91	{10, 15, 20, 25, 30, 35}	,,	,, ,,	29
2/8/16	7.15 ,,	Polaris	2·2	70	{10, 15, 20, 25, 30, 35, 40, 45, 50}	Medium	All visible	30
19/8/16	5.15 ,,	,,	2·2	70	5,7,9,11	,,	,,	33
19/8/16	5.20 ,,	,,	2·2	91	{120, 100, 80, 40, 30}	,,	40 & 30 not visible	33
30/8/16	2.15 ,,	α Ursæ Maj.	2·0	70	{5, 7, 9, 11, 13}	Good	All visible	35
21/9/16	12.5 ,,	Moon	...	70	5, 7, 10	Clouds	Clear	38
21/9/16	12.10 ,,	Venus	...	70	3	,,	,,	38
21/9/16	12.15 ,,	,,	...	91	30	,,	,,	38
26/10/16	10.40 a.m.	ψ Ursæ Maj.	3·2	70	20, 25, 30	,,	Faint	43
26/10/16	12.50 p.m.	γ ,,	2·5	70	15, 20	,,	,,	43

The images are clear and well defined, as indeed they should be, being taken in almost monochromatic light. It is, of course, essential to set the plate in focus with the filter in position.

Unfavourable weather unfortunately prevented any attempts to photograph Regulus when near the sun, though this star would be very well adapted for testing Einstein's theory.* In a really clear sky, however, there should be no particular difficulty, as the scattered light is proportional to $1+\cos^2\beta$, *i.e.* it can only be twice as great near the sun as at 90 degrees from it. There seems to be every reason to hope that stars even fainter than the first and second magnitude could be photographed quite near the sun

* Venus and Mercury might be almost equally advantageous at certain periods.

in a fine climate, *e.g.* in Kashmir or perhaps at Mount Wilson. It would be most desirable that these experiments be carried through by some observatory which possesses a suitable instrument,* and which is favoured by really clear air.

In spite of the increased exposure attending the use of filters, their value in stellar photography may be considerable. Thus, for instance, it may be possible to photograph the comparatively cold companions of some double stars by using a red filter, whereas the brighter star prevents this if no filter is employed. It may even be possible to discover dark stars if sufficiently red and infra-red sensitive plates, such as those described by Sir William Abney, are employed.

The use of these light filters is often of great advantage for observing. In really good weather the image one observes is apparently the true diffraction disc, as is proved by the fact that one can separate double stars almost up to the theoretical limiting power of the telescope. It is for this reason that it is advisable to choose a magnifying power such that the image of the star just covers one cone in the eye; this is about half the diameter of the objective measured in millimetres. The image of the star subtends an angle equal to $2{\cdot}44\frac{\lambda}{D}$, so that the solid angle is $4{\cdot}68\frac{\lambda^2}{D^2}$. The light from this solid angle of blue sky is

$$\frac{4{\cdot}72\,.\,10^{-19}}{\lambda^2 D^2}\,.\,\frac{1+\cos^2\beta}{\cos\gamma}\,I,$$

whereas the light from an nth magnitude star is $I\,.\,2{\cdot}51^{-(26{\cdot}6+n)}$. The two are equal when

$$n = 5\log\lambda + 5\log D - 2{\cdot}5\log\frac{1+\cos^2\beta}{\cos\gamma} + 19{\cdot}2.$$

Putting $\lambda = 5000$ Å., $D = 15$ cm., and $\frac{1+\cos^2\beta}{\cos\gamma} = 2$, the light from the star is equal to the light from the sky when $n = 2{\cdot}8$. As one can probably see a fifth magnitude star in daytime through a six-inch telescope, it would seem as if one could distinguish a bright point on an evenly illuminated surface when the amount of light falling upon one cone is about 10 per cent. greater than the amount falling upon all the others. It is obvious that the contrast may be improved by using greater wave-lengths, though of course the eye imposes a limit at the red end of the spectrum.† The curves using normal sensitiveness for the eye are drawn in fig. 6, but they can of course only be applied to give a rough idea, as the constant of Fechner's law varies with the wave-length by a different amount for each individual eye.

A fact which these experiments brought out very clearly was

* Probably the object-glass need not be achromatised.

† In certain parts of the sky considerable advantage may be gained by using a Nicol prism, either with or without a filter.

the enormous difference in the amount of scattered light from apparently blue sky on different days. It is clear that particles of dust or moisture, whose size is not small in comparison to a wave-length, will not conform to Rayleigh's λ^{-4} law. The amount of red light scattered may thus be used for constructing an instrument to give a quantitative measure of the transparency of the air. All that is necessary is to construct an instrument in which the sky can be compared photographically or visually with some known artificial light. A convenient form which might be developed into a standard instrument consists in a vertical camera for photographing the zenith, which enables a strip of sky forming an image some 10 cm. long and 2 cm. broad to be focussed on the plate. (In a standard instrument a pin-hole of well-defined dimensions

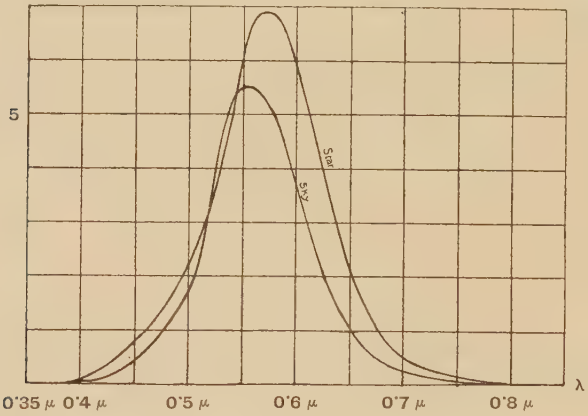

FIG. 6.

might be used.) Next to the strip formed by the light from the sky a strip is exposed to the radiation of a tungsten-filament lamp, burning at a definite voltage and producing a known amount of light. A shutter is pulled across both strips at a constant rate such that the one end of each is slightly overexposed, whilst the other ends are not exposed at all. The current through the lamp is interrupted when the shutter reaches its full extension, so that the blackening of both strips diminishes gradually to zero. All light, of course, passes through standard filters. The point on the two strips at which the blackening is equal defines the amount of light of the wave-length determined by the filter, which is scattered by the air. The zenith must, of course, be photographed at such a time that the sun is some definite distance from it, say 45 degrees. An instrument of this type would enable the transparency of the air at widely different times and localities to be compared quantitatively, at any rate in the daytime. The steadiness of the air must, of course, be measured as heretofore by observing or photographing the apparent movement of a star. Quite apart from the

astronomical value of being able to compare different atmospheric conditions quantitatively, the transparency of the air to radiation may be of considerable interest in various meteorological investigations. As radiation is supposed to be the most important factor in producing equilibrium in the upper air, any measurements which have a bearing on this should be of value. It is probable, however, that the haze and dust are usually at lower levels, in which convection quite overpowers radiation. Only in those cases in which minute crystals of ice are the cause of scattering would the quantities interest meteorologists, but it is not improbable that the polarisation of different parts of the sky might be used to distinguish such days.

Summary.

Methods are indicated for testing by photography Einstein's hypothesis of the refraction of light by a gravitational field.

The most favourable conditions for photographing stars in the daytime are discussed. Experiments are described showing that stars may be photographed in daytime by using red filters and red sensitive plates, and their value for observing stars during the day is emphasised.

An instrument is described by means of which the transparency of the air at different times and places may be compared quantitatively.

It is suggested that experiments on daylight photography of stars be undertaken by some observatory possessing a suitable instrument, and enjoying a fine climate, with a view to testing Einstein's theory.

Sidmouth:
1916 *Dec.* 4.

The Choice of an Origin for Galactic Longitudes.

By C. D. Perrine, Assoc. R.A.S.

The increasing use of galactic co-ordinates renders desirable the consideration of the best origin for the longitudes.

Innes has adopted for his tables of conversion * the position of the solar apex. The choice of a fixed point as origin is, as he points out, preferable to one which is variable. It is doubtful, however, if the effect of precession would be large enough for a long period to make any great difference in many of the problems for which galactic co-ordinates are used. As the basis, however, for a system of co-ordinates for accurate star positions, the case is different. The choice of any natural origin, such as the solar apex, is fraught with the uncertainties inherent to the determination of

* *Union Obs. Circular*, No. 29.

its exact position, which may be quite as great as the effects of precession. In the choice of any such origin it will be necessary to be more or less arbitrary, as Innes has been.

The object of this note is to raise the question whether there is not a more convenient and logical origin than the solar apex. Recent investigations have shown that there is a range of something like 60° or more between the positions of the apex as derived from different groups of stars, and that these discordances appear to be systematic. It appears, therefore, not to be as fixed as has been assumed. There is the further objection that the accepted position of the solar apex is at some distance from the galactic plane. This is not serious, but it is preferable that the origin lie in or close to the fundamental plane.

It is, I think, constantly becoming more clear that the so-called ellipsoidal axis has great significance in our stellar system. While the positions found for it vary somewhat, and it will probably prove to be variable to some extent, there seems to be little doubt that it does, in fact, lie in the galactic plane. Altogether it seems to me to offer peculiar advantages, and I would, therefore, propose the adoption of the southern end of that axis, the extremity nearest to the present assumed apex of solar motion, as the origin from which to reckon galactic longitudes.

As the vertex

$$\alpha \quad 274^\circ$$
$$\delta \quad -12^\circ$$

found by Eddington * from the Boss stars lies almost exactly in the galactic plane (as determined by Newcomb), this would seem to be a most natural position to adopt, assuming the epoch 1900·0 where necessary.

Observatorio Nacional Argentino,
Cordoba:
1916 *October* 16.

Photographic Determination of the Parallax of Three Southern Binary Systems. By J. Voûte.

(*Communicated by H.M. Astronomer at the Cape of Good Hope.*)

The plates for these determinations were taken with the Cape astrophotographic refractor during the years 1914, '15, '16. The observations were made with the view of finding whether a longer list of stars might be taken in hand with that instrument after the completion of the parallax observations made with the Transit Circle. The results obtained from instruments with short focus

* *Stellar Movements*, p. 102.

are not without value, as was shown by Kapteyn in his reductions of the Donner plates and also by Kostinsky's parallax work. The results obtained for these three stars having proved satisfactory, a list of twenty-three stars has been taken in hand.

On every plate exposures are recorded at two consecutive epochs of maximum parallactic factor. In general three exposures of five minutes were made at each epoch, the star-images being shifted in declination. A longer exposure than that found necessary to obtain measurable images of faint stars was chosen in order to reduce the effect of a guiding error. All plates were taken within an hour before the meridian transit of the parallax-star, and as far as possible the second exposure was made at the same hour-angle.

A rotating sector for reducing the images of the bright stars was not available, and consequently the proposal suggested by Grossmann was adopted, viz. of fixing in front of the object-glass a wire (in this case a 2·4 mm. watch-spring) in the direction of the parallel, thus producing a diffraction figure in the direction of the declination. Without this wire it would be impossible to point accurately on the images of a bright double star such as α Crucis, while pointings on the first order diffraction figures may be made with sufficient accuracy.

The plates have been measured with the measuring arrangement of the Zeiss stereo-comparator by a method which will be described in a future paper, where a description of the results derived from the complete list of stars will be published.

κ Toucani System.—The two stars κ Toucani and Lac. 353 form a physical system, as evidenced by the agreement of their proper motions, and, as is now confirmed, also of their parallaxes. Both stars are binaries; the component of κ Toucani has been observed through an arc of about 15°, and that of Lac. 353 about 85°; their periods are therefore still unknown.

The mean magnitude of the eleven comparison stars used for determining the plate constants is 10·4.

From this region four plates were obtained, every set of two consecutive plates giving a value of the parallax after eliminating the proper motions.

The two independent determinations are:

	Plates		Relative π.
	a and *b*.	*c* and *d*.	
κ Toucani	+0″·16	+0″·17	+0″·165
Lac. 353	+0·19	+0·11	+0·150

Hence the parallax of the system relative to that of the comparison stars is +0″·157.

Hertzsprung has suspected a hypothetical parallax of 0″·09 (*A.N.*, **190**, 113).

α Crucis System.—The three stars forming this system are

bright stars α_1 and α_2 Crucis accompanied by Lac. 5147. According to Boss the three stars have nearly the same P.M.

Four plates were obtained of this region.

The mean magnitude of the ten comparison stars is 9·6.

The two independent determinations are:

	Plates		Relative π.
	a and *c*.	*b* and *d*.	
	″	″	″
α_1 and α_2 Crucis	+0·01	−0·01	0·000
Lac. 5147	+0·01	−0·01	0·000

Hence the relative parallax of the system is 0″·000.

Gill's value for the parallax of α_1 and α_2 Crucis is +0″·055 (absolute).

Boss 4386 = *Melbourne* 4 = β 416.—This is one of the most interesting systems. The tenth magnitude component, which is at a distance of 34″ from the bright star, was found by Herschel; since discovery it has described only a few degrees in its orbit; the other component found in Melbourne, rediscovered by β in 1876, has a period of forty-three years. The exposures, taken under splendid condition of definition, show an elongation in the β-pair, the Herschel star being quite isolated, so that the parallax can be obtained for both images.

Six plates were obtained.

The mean magnitude of the ten comparison stars is 10·0.

The three independent determinations are:

	Plates			Relative π.
	a and *b*.	*c* and *e*.	*d* and *f*.	
	″	″	″	″
β 416	+0·18	+0·20	+0·14	+0·173
h's star	+0·12	+0·18	+0·18	+0·160

Hence the relative parallax of the system is +0″·166.

The hypothetical parallax computed by Luplau-Janssen is +0″·182 (*A.N.*, **202**, 5).

Royal Observatory,
Cape of Good Hope:
1916 *Oct.* 27.

On Einstein's Theory of Gravitation, and its Astronomical Consequences. By W. de Sitter, Assoc. R.A.S. Second Paper.*

Contents of Second Paper.

20. Field of n moving bodies: the differential equations.
21. First method of integration.
22. Second method of integration: comparison of the two methods.
23. Problem of n bodies: equations of motion. Planetary theories.
24. Lunar theory: equations of motion.
25. Further development, perturbing forces.
26. Motion of lunar perigee and node: comparison with observations.
27. Field of moving sun and motion of infinitesimal planet in it, compared with field of sun at rest.
28. Gravitational field of the system of the fixed stars. Effect on spectral lines. Upper limit for total mass of stars.
29. Irreality of co-ordinates. Relativity of rotation.
30. On the origin of inertia. Hypothesis of distant masses. Conclusion.

20. We will now derive the gravitational field of n bodies, which have arbitrarily given motions relatively to the system of co-ordinate axes. We will suppose the bodies to be rigid spheres. By this we mean that in a system of co-ordinates relatively to which any one body is at rest, its outer surface would be described as a sphere. In a system relatively to which it moves it then undergoes the well-known Lorentz-contraction. By the attribute "rigid" I mean that the form of the bodies will not be influenced by their mutual gravitation. We further suppose that their dimensions are small compared with their mutual distances, and also their velocities are supposed to be small compared with the velocity of light. The different bodies will be distinguished by indices μ or ν; the indices i and j refer to the co-ordinate axes as before.

We will take rectangular Cartesian co-ordinates, $x_1 = x$, $x_2 = y$, $x_3 = z$, and $x_4 = ct$, t being the time. The g_{ij} will be

$$\begin{aligned} g_{ii} &= -1 + \gamma_{ii}, \quad & g_{ij} &= \gamma_{ij}, \\ g_{i4} &= \chi_i, & g_{44} &= 1 + \gamma, \end{aligned} \qquad (i, j = 1, 2, 3).$$

The γ_{ij} and γ will be of the first order, χ_i will be of the order $\frac{3}{2}$. Only in γ we require the terms of the second order.

We must now form the equations (first paper, p. 706)

$$(\text{II.}) \qquad G_{ij} = -\kappa T_{ij} + \tfrac{1}{2}\kappa g_{ij} T.$$

* See first paper, *M.N.*, vol. lxxvi. p. 699. For this second paper I have, beyond the help from my colleagues and friends already acknowledged in the first paper, had the great privilege of several conversations with Professor Einstein himself, during a visit of his to Leiden.

To the list of papers given in the footnote to page 700 must be added:

VII. Einstein, "Hamiltonsches Prinzip und allgemeine Relativitätstheorie," *Sitzungsber. Berlin*, 1916 Oct., p. 1111.

To the required order of accuracy we have

$$(63)\qquad \begin{cases} T_{ij} = \rho \dot{x}_i \dot{x}_j, & T_{i4} = -\rho \dot{x}_i, \\ T_{44} = \rho(1 + \gamma + \phi^2), & T = \rho, \end{cases}$$

where we have again put

$$\dot{x}_i = \frac{dx_i}{cdt}, \qquad \phi^2 = \sum_a{}' \dot{x}_a^2,$$

The velocities $\dot{x}_i$ are of the order $\frac{1}{2}$. We have neglected all terms derived from the inner constitution of the gravitating bodies. By the same reasoning as in art. 10 (first paper) we can show that they can be taken into account by slightly altering the meaning of the symbol ρ.

We will suppose that γ_{ij}, χ_i, and γ do not depend explicitly on $x_4 = ct$, but only through the co-ordinates x_1, x_2, x_3 of the n bodies, which are functions of t. Consequently, a differentiation with respect to x_4 increases the order of smallness by $\frac{1}{2}$, and such a term as, *e.g.*, $\frac{\partial^2 \gamma}{c^2 \partial t^2}$, is of the second order. To the required order of accuracy the equations are found to be

$$(64)\qquad \begin{cases} G_{ij} = \frac{1}{2}\dfrac{\partial^2 \gamma}{\partial x_i \partial x_j} - \frac{1}{2}\displaystyle\sum_a{}' \frac{\partial^2 \gamma_{aa}}{\partial x_i \partial x_j} + \frac{1}{2}\sum_a{}' \left[\frac{\partial^2 \gamma_{ia}}{\partial x_a \partial x_j} + \frac{\partial^2 \gamma_{ja}}{\partial x_a \partial x_i} - \frac{\partial^2 \gamma_{ij}}{\partial x_a^2} \right] = -\frac{1}{2}\delta_{ij}\kappa\rho, \\ G_{i1} = -\frac{1}{2}\displaystyle\sum_a{}' \frac{\partial^2 \gamma_{aa}}{\partial x_i c \partial t} + \frac{1}{2}\sum_a{}' \left[\frac{\partial^2 \gamma_{ia}}{\partial x_a c \partial t} + \frac{\partial^2 \chi_a}{\partial x_i \partial x_a} - \frac{\partial^2 \chi_i}{\partial x_a^2} \right] = \kappa\rho\dot{x}_i, \\ G_{44} = -\frac{1}{2}\displaystyle\sum_a{}' \frac{\partial^2 \gamma}{\partial x_a^2} - \frac{1}{2}\sum_a{}' \frac{\partial^2 \gamma_{aa}}{c^2 \partial t^2} + \sum_a{}' \frac{\partial^2 \chi_a}{\partial x_a c \partial t} + \frac{1}{4}\sum_a{}' \left(\frac{\partial \gamma}{\partial x_a} \right)^2 \\ \qquad + \frac{1}{4}\displaystyle\sum_a{}' \sum_\beta{}' \frac{\partial \gamma}{\partial x_a} \frac{\partial \gamma_{\beta\beta}}{\partial x_\beta} - \frac{1}{2}\sum_a{}' \sum_\beta{}' \frac{\partial}{\partial x_a} \left[\gamma_{a\beta} \frac{\partial \gamma}{\partial x_\beta} \right] = -\frac{1}{2}\kappa\rho(1 + \gamma + 2\phi^2). \end{cases}$$

It will be remembered that $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$, and that Σ' denotes a sum over the values 1, 2, 3 only.

If we neglected all but the terms of the lowest order, the equations would be linear. Then the field of the n bodies would be got by superposing the fields of the separate bodies, each computed as if the others did not exist. The field of a single body has been found in art. 11. Thus, if we take the system B of art. 11 we find that the γ_{ij} must be of the form

$$\gamma_{ii} = \gamma, \quad \gamma_{ij} = 0 \text{ for } i \neq j.$$

21. If this is introduced into the equations for G_{ij} and the terms of the first order of G_{44}, these equations all reduce to

$$(65)\qquad \sum_a{}' \frac{\partial^2 \gamma}{\partial x_a^2} = \kappa\rho.$$

The integral is

$$\gamma = -\frac{\kappa}{4\pi} \int \frac{\rho dV}{r}.$$

Here r is the three-dimensional distance from the point (x_1, x_2, x_3) at which the value of γ is required to the point (x_1', x_2', x_3') at which the density is ρ. Thus $r^2 = \sum_a (x_a - x_a')^2$. Further, dV is the three-dimensional element of volume. The integral must be taken over the whole of three-dimensional space. Outside the n bodies, however, $\rho = 0$, and as the bodies are supposed never to coincide, the integral consists of n separate integrals, each over one of the bodies. Since the bodies are small, we can take for r the distance to any point within the body, say the centre. In performing the integration $\int \rho dV$ we must remember that the body moves with the velocity ϕ_ν. In a system of co-ordinates in which the body would be at rest $\int \rho dV$ would be the mass. Now, however, we have in consequence of the Lorentz-contraction *

$$\int_\nu \rho dV = m_\nu \sqrt{1 - \phi_\nu^2}.$$

Therefore, for points outside the n bodies,

$$\gamma = -2\lambda^2 \sum_\nu \frac{m_\nu}{r_\nu}(1 - \tfrac{1}{2}\phi\nu^2). \tag{66}$$

Here we have put, as before, $\kappa = 8\pi\lambda^2$ and $r_\nu^2 = \sum_a (x_a - x_{a\nu})^2$, $x_{1\nu}, x_{2\nu}, x_{3\nu}$ being the co-ordinates of the body with the index ν, m_ν its mass, and ϕ_ν its velocity. For points inside one of the bodies we shall have the same value in which the sum is to be taken over the $n - 1$ other bodies, the term relating to the body itself being replaced by a term of the form (29) [first paper, art. 11].

The last term in (67) is of the second order. It will be combined below with other terms of the second order, and we will put

$$\begin{cases} \gamma = \gamma_0 + \gamma', \\ \gamma_0 = -2\lambda^2 \sum_\nu \dfrac{m_\nu}{r_\nu}. \end{cases} \tag{67'}$$

The second order term γ' is a sum of several terms, one of which is thus

$$\gamma_1' = \lambda^2 \sum_\nu \frac{m_\nu \phi_\nu^2}{r_\nu}.$$

* For any transformation $\sqrt{g}\, d\tau$ is invariant, if $d\tau$ is the four-dimensional element of volume. Thus for the three-dimensional element of volume we have

$$\left(\frac{dV'}{dV}\right)^2 = \left(\frac{dt}{dt'}\right)^2 \frac{g}{g'}.$$

If we use a Lorentz-transformation, then $g = g'$, and to the required order $\frac{dt}{dt'} = \sqrt{(1 - \phi\nu^2)}$, therefore $dV' = dV\sqrt{(1 - \phi\nu^2)}$, where the letters without accents refer to the system in which the body is at rest.

We now use this value of γ in the equation for G_{i4}, which then becomes

$$\sum_a{}' \frac{\partial}{\partial x_a}\left[\frac{\partial \chi_a}{\partial x_i}-\frac{\partial \chi_i}{\partial x_a}\right]=2\kappa\rho\dot{x}_i+2\frac{\partial^2\gamma_0}{\partial x_i c\partial t}.$$

In the partial differentiation of γ_0 with respect to t, we must consider $x_{i\nu}$ as a function of t, thus $\frac{\partial x_{i\nu}}{c\partial t}=\dot{x}_{i\nu}$, but x_i does not depend explicitly on t. The integral is found to be

$$\chi_i=\frac{\kappa}{2\pi}\int\frac{\rho\dot{x}_i}{r}d\mathrm{V}=4\lambda^2\sum_\nu\frac{m_\nu\dot{x}_{i\nu}}{r_\nu} \tag{68}$$

The correction for the Lorentz-contraction can here be neglected, as it would give a term of the order $\frac{5}{2}$. From (68) we find

$$\sum_a{}'\frac{\partial^2\chi_a}{\partial x_a c\partial t}=2\frac{\partial^2\gamma_0}{c^2\partial t^2}.$$

This we introduce into the equation G_{44}. We then divide by $(1+\gamma)$, and subtract (65). Then we find

$$\sum_a{}'\frac{\partial^2(\gamma'-\gamma_1')}{\partial x_a{}^2}=2\kappa\rho\phi^2+\frac{\partial^2\gamma_0}{c^2\partial t^2}+\sum_a{}'\left(\frac{\partial\gamma_0}{\partial x_a}\right)^2. \tag{69}$$

The first term on the right gives

$$\gamma_2'=-4\lambda^2\sum_\nu\frac{m_\nu\phi_\nu{}^2}{r_\nu},$$

which can be combined with γ_1'. The second term gives

$$\gamma_3'=-\lambda^2\sum_\nu m_\nu\frac{\partial^2 r_\nu}{c^2\partial t^2},$$

The last term can, by means of (65), be transformed to

$$\sum_a{}'\frac{\partial^2}{\partial x_a{}^2}(\tfrac{1}{2}\gamma_0{}^2)-\kappa\rho\gamma_0,$$

We have thus

$$\gamma_4'=\tfrac{1}{2}\gamma_0{}^2,$$

and

$$\sum_a{}'\frac{\partial^2\gamma_5'}{\partial x_a{}^2}=-\kappa\rho\gamma_0.$$

Now whenever ρ differs from zero, say for a point within the body ν, γ_0 is of the form

$$\gamma_0=\gamma_{0\nu}-2\lambda^2\sum_{\mu\neq\nu}\frac{m_\mu}{r_\mu},$$

The first part does not depend on the other bodies; for any one point it is a constant, and it can thus be taken into account at

once in (65) by writing $\rho' = \rho(1 - \gamma_{0\nu})$ for ρ. We will suppose that this has been done, and then we will call ρ' again ρ. There remains

$$\sum_{\alpha}{}' \frac{\partial^2 \gamma_5'}{\partial x_\alpha^2} = 2\lambda^2 \kappa \rho \sum_{\mu \neq \nu} \frac{m_\mu}{\Delta_{\mu\nu}},$$

where $\Delta_{\mu\nu}$ has been written for r_μ, since the latter is required only for points within the body ν. We then find

$$\gamma_5' = -4\lambda^2 \sum_{\nu} \frac{m_\nu}{r_\nu} \sum_{\mu \neq \nu} \frac{m_\mu}{\Delta_{\mu\nu}}.$$

Collecting results we have

$$(70) \quad \begin{cases} \gamma_{ii} = \gamma_0, \quad \gamma_{ij} = 0, \\ \gamma_0 = -2\lambda^2 \sum_{\nu} \frac{m_\nu}{r_\nu}, \\ \chi_i = 4\lambda^2 \sum_{\nu} \frac{m_\nu \dot{x}_{i\nu}}{r_\nu}, \\ \gamma = \gamma_0 + \gamma', \\ \gamma' = \frac{1}{2}\gamma_0^2 - 3\lambda^2 \sum_{\nu} \frac{m_\nu \phi_\nu^2}{r_\nu} - 4\lambda^2 \sum_{\nu} \frac{m_\nu}{r_\nu} \sum_{\mu \neq \nu} \frac{m_\mu}{\Delta_{\mu\nu}} - \lambda^2 \sum_{\nu} m_\nu \frac{\partial^2 r_\nu}{c^2 \partial t^2}. \end{cases}$$

If we put

$$(71) \quad \sum_{\alpha}{}' (x_{i\nu} - x_i)\dot{x}_{i\nu} = r_\nu \epsilon_\nu, \quad \sum_{\alpha}{}' (x_{i\nu} - x_i)\ddot{x}_{i\nu} = r_\nu \eta_\nu,$$

and if by P_ν we denote the angle at the body ν between the velocity of that body and the radius r_ν, then we have

$$\epsilon_\nu = -\phi_\nu \cos P_\nu$$

and

$$(72) \quad \frac{\partial^2 r_\nu}{c^2 \partial t^2} = \frac{\phi_\nu^2 - \epsilon_\nu^2}{r_\nu} + \eta_\nu.$$

22. The analysis of the preceding article is due to Dr. J. Droste.* Practically simultaneously Einstein † has given another solution of the same equations. Einstein does not introduce the condition that $\dot{x}_i$ shall be small; on the other hand, he gives his solution only to the first order in κ. It is, however, easy to complete it by adding the terms of the second order in g_{44}. Einstein takes as co-ordinates $x_1 = x$, $x_2 = y$, $x_3 = z$, $x_4 = ict$ $(i = \sqrt{-1})$. Then he puts

$$g_{pq} = -\delta_{pq} + \gamma_{pq}.$$

* J. Droste, *Proceedings Academy of Sciences at Amsterdam*, June 1916, (vol. xix. p. 447).

† Einstein VI., *Sitzungsber. Berlin*, June 1916.

Thus for i and j different from 4 these γ_{ij} are the same as those in the preceding article. Further,

$$\gamma_{p4} = i\chi_p, \quad \gamma_{44} = -\gamma.$$

Einstein then puts

$$\gamma_{pq} = \gamma_{pq}' + \delta_{pq}\psi, \tag{73}$$

and he introduces the conditions

$$\sum_\alpha \frac{\partial \gamma_{p\alpha}'}{\partial x_\alpha} = 0 \quad \text{or} \quad \sum_\alpha \frac{\partial \gamma_{p\alpha}}{\partial x_\alpha} = \frac{\partial \psi}{\partial x_p}.$$

The function ψ which has been arbitrarily introduced requires one more condition, which is

$$\psi = -\tfrac{1}{2}\sum_\alpha \gamma_{\alpha\alpha}'. \tag{74}$$

The equations then become, neglecting terms of the second order,

$$\sum_\alpha \frac{\partial^2}{\partial x_\alpha^2}\Big(\gamma_{pq}' - \tfrac{1}{2}\delta_{pq}\sum_\beta \gamma_{\beta\beta}'\Big) = 2\kappa\Big(T_{pq} - \tfrac{1}{2}\delta_{pq}\sum_\beta T_{\beta\beta}\Big), \tag{75}$$

or

$$\sum_\alpha \frac{\partial^2}{\partial x_\alpha^2}\gamma_{pq}' = 2\kappa T_{pq};$$

or, if we return to the variable $ct = -ix_4$,

$$\sum_\alpha \frac{\partial^2 \gamma_{pq}'}{\partial x_\alpha^2} - \frac{\partial^2 \gamma_{pq}'}{c^2 dt^2} = 2\kappa T_{pq}. \tag{76}$$

The integral is

$$\gamma_{pq}' = -\frac{\kappa}{2\pi}\int \frac{T_{pq}'}{r'} dV. \tag{77}$$

Here $r'^2 = \sum_\alpha'(x_\alpha - x_\alpha')^2$, T_{pq}' is the value at the point (x_1', x_2', x_3') *and at the time* $t - \frac{r'}{c}$. For p and q different from 4, T_{pq}' are the same as T_{pq} in the preceding article. Further,

$$T_{p4}' = -iT_{p4}, \quad T_{44}' = -T_{44}.$$

The integration in (77) must be performed with care, the time $t - \frac{r'}{c}$ being different for each point in any of the bodies. The space over which the integral must be extended—*i.e.* in which ρ differs from zero—therefore does not coincide with the space occupied by the body at any particular time. For r′ we can take

its value for the centre of the body. Further, we find,* neglecting quantities of the order of the square of the velocity,

$$\int_\nu \rho d\mathrm{V} = \frac{m_\nu}{1+\epsilon_\nu}.$$

Therefore we have, correct to the order $\frac{3}{2}$,

$$\gamma_{pq}' = 0, \quad \gamma_{p4}' = -4i\lambda^2 \sum_\nu \frac{m_\nu \dot{x}_{i\nu}}{\mathrm{r}_\nu'},$$

$$\gamma_{44}' = 4\lambda^2 \sum_\nu \frac{m_\nu}{\mathrm{r}_\nu'}(1-\epsilon_\nu).$$

Consequently,

$$(78) \qquad \begin{cases} \psi = -\frac{1}{2}\gamma_{44}' = -2\lambda^2 \sum_\nu \frac{m_\nu}{\mathrm{r}_\nu'}(1-\epsilon^\nu), \\ \gamma_{ii} = \psi, \ \gamma_{ij} = 0, \ \gamma = -\gamma_{44} = \psi, \\ \chi_i = 4\lambda^2 \sum_\nu \frac{m_\nu \dot{x}_{i\nu}}{\mathrm{r}_\nu'}(1-\epsilon_\nu). \end{cases}$$

It will be seen that this solution differs from (70)—if there also we neglect the terms of the second order—by the substitution of r_ν' for r_ν and by the factor $(1-\epsilon_\nu)$. Now, to the first order in the velocities we have

$$\mathrm{r}' = \mathrm{r} - \tfrac{1}{2}\mathrm{r}' \frac{\partial \mathrm{r}}{c\partial t} = \mathrm{r}(1-\epsilon).$$

Consequently, the factor $(1-\epsilon)$ is exactly cancelled, and (78) agrees with (70), neglecting only the terms of the second order (and in χ_i of the order $\frac{5}{2}$). We could now add the terms of the second order in the same way as was done in the preceding article, and we should find the same result, with the exception of the term γ_3', which is here included in the first approximation. The two systems may differ in the terms of the second order owing to the difference in the additional conditions introduced to fix the system of reference.

In both systems the velocity of light is

$$v = c\left[1 + \gamma + 4\lambda^2 \sum_\nu \frac{m_\nu \phi_\nu}{\mathrm{r}_\nu} \cos \mathrm{V}_\nu\right],$$

where V_ν is the angle between the element $d\sigma$ of the ray of light (taken positive in the direction of the propagation of light) and the velocity of the body ν.

In the system (78) the g_{ij} are functions of the distance r' between the point at which gravitation acts at the time t and the body from which it emanates at the time $t - \frac{\mathrm{r}'}{c}$. This shows that

* See Lorentz, *Theory of Electrons*, note 19 (p. 254), where $\frac{\mathrm{V}_r}{c} = -\epsilon$.

gravitation is propagated with the velocity of light. Nevertheless, in the system (70) the g_{ij} depend only on the distance r between the *simultaneous* positions of the point acted upon and the source of the action, and we should apparently be forced to make the conclusion that the velocity of propagation is instantaneous. The paradox is, however, only apparent. If we speak of propagation of gravitation at all, we must conceive the g_{ij} as variable with the time, and in the given problem this variability can only arise from the motions of the bodies producing the gravitational field. Consequently, by the method of art. 21, where these velocities are assumed to be small, and third differentials with respect to the time are neglected entirely, it is *a priori* impossible to derive anything at all about the velocity of propagation.

23. In the planetary theories we wish to determine the motion of one of the n bodies in the field produced by the others and itself. If the body is of finite dimensions, the accelerations impressed on its different points will not be the same. The body being rigid, its points must, however, all have the same motion. There are thus produced strains in it which affect the T_{ij}, and through these again the gravitational field. These effects are, of course, of the second order and, if the dimensions of the bodies are small compared with their mutual distances, they will be negligible. Then the equations of motion of a body are the same as those of any point in it, say the centre. The equations can be written in any of the forms given in the first paper. We will use the formulas (47), and for χ_i and γ we use the values (70).

If the body whose motion is considered has the index μ, then in γ_0 we must leave out the term with that index. The same remark applies to the single sums in χ_i and in the second and fourth terms of γ'. In the double sum, which forms the third term of γ', we must in the first sum omit the body μ, but in the second (inner) sum it must be included. We will bring the term so arising into evidence by taking it out of the sum. Then the index μ must be excluded from *all* sums. The complete equation then becomes:

$$
\begin{aligned}
(82)\quad \frac{d^2x_{i\mu}}{c^2dt^2} &+ \lambda^2 \sum_\nu m_\nu \frac{x_{i\mu}-x_{i\nu}}{r_\nu^3} = 4\lambda^4 \sum_\nu \sum_\kappa m_\nu m_\kappa \frac{x_{i\mu}-x_{i\nu}}{r_\nu^3 r_\kappa} \\
&+ 4\lambda^2 \sum_\nu m_\nu {\sum_p}' \frac{x_{p\mu}-x_{p\nu}}{r_\nu^3}\dot{x}_{i\mu}\dot{x}_{p\mu} \\
&- \lambda^2 \sum_\nu m_\nu \frac{x_{i\mu}-x_{i\nu}}{r_\nu^3}\{\phi_\mu^2 + 2\phi_\nu^2 - \tfrac{3}{2}\epsilon_\nu^2 + \tfrac{1}{2} r_\nu \eta_\nu\} \\
&+ \lambda^2 \sum_\nu m_\nu \left\{ \frac{7}{2}\frac{\ddot{x}_{i\nu}}{r_\nu} + 3\frac{\dot{x}_{i\mu}-\dot{x}_{i\nu}}{r_\nu^2}\epsilon_\nu + 4{\sum_p}' \frac{(x_{i\mu}-x_{i\nu})\dot{x}_{p\nu} - (x_{p\mu}-x_{p\nu})\dot{x}_{i\nu}}{r_\nu^3}\dot{x}_{p\mu} \right\} \\
&- 2\lambda^4 \sum_\nu m_\nu \frac{x_{i\mu}-x_{i\nu}}{r_\nu^3} \sum_{\kappa\neq\nu} \frac{m_\kappa}{\Delta_{\kappa\nu}} - 4\lambda^4 m_\mu \sum_\nu m_\nu \frac{x_{i\mu}-x_{i\nu}}{r_\nu^4}.
\end{aligned}
$$

It should be noted that r_ν is identical with $\Delta_{\mu\nu}$.

The left-hand members put equal to zero give the motion according to Newton's law of gravitation. The first three terms on the right are the generalisation of the terms already found in the first paper for the motion of an infinitesimal planet.

In applying this equation to the solar system we must take account of the smallness of most of the masses. The sun will be given the index $\mu = 0$.

We will introduce co-ordinates relatively to the sun by putting

$$x_i' = x_i - x_{i_0},$$

from which

$$\frac{d^2x_i'}{c^2dt^2} = \frac{d^2x_i}{c^2dt^2} - \frac{d^2x_{i_0}}{c^2dt^2}.$$

In the right-hand member we must then put

$$\dot{x}_i = \dot{x}_i' + \dot{x}_{i_0}, \quad \ddot{x}_i = \ddot{x}_i' + \ddot{x}_{i_0}.$$

In these right-hand members we can use the values given by the Newtonian theory. The system of co-ordinates to which (82) is referred was fixed in art. 20 by the additional conditions introduced during the integration. One of these was that at a large distance from all masses the geodetic lines are straight lines. In Newtonian mechanics this condition is satisfied relatively to a system of co-ordinates of which the origin is in the centre of gravity of all masses. Thus for the Newtonian values we will have to take

$$\dot{x}_{i_0} + m_\mu \dot{x}_{i_\mu} + \sum_\nu m_\nu \dot{x}_{i_\nu} = 0,$$

or, if we put

$$M = 1 + m_\mu + \sum_\nu m_\nu,$$

$$\dot{x}_{i_\mu} = \left(1 - \frac{m_\mu}{M}\right)\dot{x}_{i_\mu}' - \sum_\nu \frac{m_\nu}{M}\dot{x}_{i_\nu}',$$

and the same for second differentials. The sums exclude the values 0 and μ. After the substitution we drop the accents.

We must now write the equation (82) first for the index μ and then for the index 0, and subtract the latter from the former. I will slightly alter the notation of the mutual distances, to bring it into harmony with the usual notation of planetary theories. In equation (82) we first replace r_ν by $\Delta_{\mu\nu}$. Then we replace

$$\Delta_{0\mu},\ \Delta_{0\nu},\ \Delta_{\mu\nu},\ \Delta_{\kappa\nu};\ m_0,\ m_\mu,\ m_\nu$$

by

$$r,\quad r_\nu,\quad \Delta_\nu,\ \Delta_{\kappa\nu};\quad 1,\quad m,\ m_\nu$$

respectively, and we also drop the index μ of the co-ordinates, and we replace λ^2 by ${\lambda_0}^2$. The masses m and m_ν are very small. We will suppose them to be of the order $\frac{1}{2}$. Then in the right-hand

members all terms beyond the first three will be of the order $2\frac{1}{2}$, and the resulting equation becomes, correct to the second order,

$$(83)\qquad \frac{d^2x_i}{c^2dt^2}+\lambda_0{}^2(1+m)\frac{x_i}{r^3}+\lambda_0{}^2\sum_\nu m_\nu\frac{x_i-x_{i\nu}}{\Delta_\nu{}^3}-\lambda_0{}^2\sum_\nu m_\nu\frac{x_{i\nu}}{r_\nu{}^3}$$
$$=4\lambda_0{}^4\frac{x_i}{r^4}+4\lambda_0{}^2\sum_p{}'\frac{x_p}{r^3}\dot{x}_p\dot{x}_i-\lambda_0{}^2\frac{x_i}{r^3}\phi^2.$$

The left-hand member gives the unperturbed motion and the perturbing forces according to Newton's law. The right-hand member contains nothing beyond the terms which have already been found in the first paper (art. 15) for an infinitesimal planet.

If we wish to take account of the terms of the order $\lambda^4 m$, then we must: 1st, in the ordinary perturbing function on the left hand use the co-ordinates increased by the "perturbations" produced by the right-hand member, *i.e.* we must use the formula (52, B) instead of Kepler's third law, and we must take account of the motion of the perihelion found in art. 16; 2nd, we must in the right-hand member of (83) use the co-ordinates increased by their Newtonian perturbations of the first order; and 3rd, we must add to this right-hand member the following terms:—

$$(83')\quad\left\{\begin{aligned}
&\lambda_0{}^2m\left\{2\frac{\dot{x}_i\dot{r}}{r^2}-\frac{x_i}{r^3}\left[4\phi^2-\tfrac{3}{2}\dot{r}^2\right]\right\}+8\lambda^4m\frac{x_i}{r^4}\\
&+\lambda_0{}^2\sum_\nu m_\nu\left\{\frac{x_i-x_{i\nu}}{\Delta_\nu{}^3}\left[4\chi_\nu+\tfrac{3}{2}\epsilon_\nu{}^2-\phi^2-2\phi_\nu{}^2\right]+\epsilon_\nu\left[3\frac{\dot{x}_i+\dot{x}_{i\nu}}{\Delta_\nu{}^2}+\frac{7\Delta_\nu\dot{x}_i}{r^3}\right]\right.\\
&\qquad\left.+\dot{x}_i\left[\frac{4\epsilon_\nu'}{\Delta_\nu{}^3}-\frac{7r_\nu\dot{r}_\nu}{r^3}\right]+6\frac{x_i}{r^3}\chi_\nu+\frac{x_{i\nu}}{r_\nu{}^3}\left[\tfrac{3}{2}\dot{r}_\nu{}^2-2\phi_\nu{}^2\right]+3\frac{\dot{r}_\nu\dot{x}_{i\nu}}{r_\nu{}^2}\right\}\\
&+\lambda_0{}^4\sum_\nu m_\nu\left\{\frac{x_i-x_{i\nu}}{4\Delta_\nu{}^3}\left[\frac{3}{r_\nu}-\frac{2r\cos\psi_\nu}{r_\nu{}^2}\right]-\frac{2x_i}{r_\nu{}^3\Delta_\nu}-\frac{x_i\cos\psi_\nu}{2r^2r_\nu{}^2}\right.\\
&\qquad\left.+\tfrac{7}{2}\frac{x_{i\nu}}{r_\nu{}^3}\left[\frac{2}{r_\nu}+\frac{1}{r}-\frac{1}{\Delta_\nu}\right]\right\},
\end{aligned}\right.$$

where for abbreviation we have put

$$\Delta_\nu\epsilon_\nu=\sum_p{}'(x_{p\nu}-x_p)\dot{x}_{p\nu},\qquad \Delta_\nu\epsilon_\nu'=\sum_p{}'(x_p-x_{p\nu})\dot{x}_p,$$
$$rr_\nu\cos\psi_\nu=\sum_p{}'x_px_{p\nu},\qquad \chi_\nu=\sum_p{}'\dot{x}_p\dot{x}_{p\nu}.$$

All these terms are, of course, extremely small, and none of them will give rise to an effect which is at all within the reach of observations.

24. For the lunar theory * I will take a system of reference which has its origin in the sun. The masses of the sun, the earth, and the moon will be m, 1, and μ respectively. We will further

* See also De Sitter, "Planetary Motion and the Motion of the Moon according to Einstein's Theory," *Proceedings Acad. Amsterdam*, June 1916 (vol. xix. p. 367).

put $\lambda_0^2 = \lambda^2 m$, $\lambda_1^2 = \lambda^2$, if λ^2 is expressed with the mass of the earth as unit. Consequently, λ_0^2 will have the same numerical value as before, and $\lambda_1^2 = \lambda_0^2/m$. We will put

x_i, Δ = heliocentric co-ordinates and distance of the moon,
ξ_i, ρ = " " " " earth,
x_i, r = geocentric " " " moon.

Further, we put

$$\sum_a' \dot{x}_a^2 = \phi^2, \quad \sum_a' \dot{x}_a^2 = \phi_1^2, \quad \sum_a' \dot{\xi}_a^2 = \phi_0^2,$$

whence

$$\phi_1^2 = \phi_0^2 + \phi^2 + 2\sum_a' \dot{x}_a \dot{\xi}_a .$$

We now must compute the difference

$$\frac{d^2 x_i}{c^2 dt^2} - \frac{d^2 \xi_i}{c^2 dt^2},$$

both differential quotients being computed by (82). The terms of (82) will be divided into three groups, viz.:—

a. Those which only contain the masses 1 and μ.

b. Those in $\frac{d^2 x_i}{c^2 dt^2}$ containing m or $m \cdot \mu$, and in $\frac{d^2 \xi_i}{c^2 dt^2}$ containing m and $m \cdot 1$,

c. The others, *i.e.* in $\frac{d^2 x_i}{c^2 dt^2}$, the terms with $m \cdot 1$, and in $\frac{d^2 \xi_i}{c^2 dt^2}$ those with $m \cdot \mu$.

The first group corresponds to the problem of the *two* bodies, earth and moon. The second group would give the perturbations by the sun if the complete field of the earth and sun were equal to the sum of the superposed fields of each of these bodies computed as if the other did not exist. The third group contains the terms which arise from the "interference" of the two fields.

a. In the first group I take, in the terms of the second order,

$$\ddot{x}_i = -\lambda_1^2 (1+\mu) \frac{x_i}{r^3} .$$

Further, I put

$$\epsilon = \sum_p' x_p \dot{\xi}_p .$$

We then find, after some reductions,

$$84) \quad \frac{d^2 x_i}{c^2 dt^2} + \lambda_1^2 (1+\mu) \frac{x_i}{r^3} = 4\lambda_1^4 (1 - \mu + 2\mu^2) \frac{x_i}{r^4} + 4\lambda_1^2 (1 + \tfrac{3}{4}\mu) \frac{\dot{x}_i \dot{r}}{r^2} - \lambda_1^2 (1 + 2\mu) \frac{x_i}{r^3} \phi^2$$
$$+ \lambda_1^2 \frac{x_i}{r^3} \left[(1+\mu) \phi_0^2 + 2\sum_p' \dot{x}_p \dot{\xi}_p + 3\lambda_1^2 \mu \frac{\dot{r}\epsilon}{r} + \tfrac{3}{2}(1+\mu) \frac{\epsilon^2}{r^2} + \tfrac{1}{2}(1-\mu) \sum_p' x_p \ddot{\xi}_p \right]$$
$$+ \lambda_1^2 \left[(1+2\mu) \frac{\dot{x}_i \epsilon}{r^3} + \tfrac{7}{2}(1-\mu) \frac{\ddot{\xi}_i}{r} \right].$$

If we took $\mu = 0$, $\dot{\xi}_i = \ddot{\xi}_i = 0$, this would become the formula for the motion of an infinitesimal planet. We can take

$$\phi_0^2 = \dot{\rho}^2 + \rho^2\dot{\theta}_0^2, \quad \ddot{\xi} = -\lambda_0^2\frac{\xi_i}{\rho^3}.$$

If we neglect the excentricity of the earth's orbit, we have

$$\dot{\rho} = 0, \quad \dot{\theta}_0 = \nu_0,$$

therefore

$$\phi_0^2 = \frac{\lambda_0^2}{\rho}.$$

The formula (84) then becomes, if we neglect in the second member the terms having μ as a factor,

$$(85) \quad \frac{d^2x_i}{c^2dt^2} + \lambda_1^2(1+\mu)\frac{x_i}{r^3} = 4\lambda_1^4\frac{x_i}{r^4} + 4\lambda_1^2\frac{\dot{x}_i\dot{r}}{r^2} - \lambda_1^2\frac{x_i}{r^3}\phi^2$$
$$+ \lambda_1^2\frac{x_i}{r^3}{\sum_p}'\left(2\dot{x}_p\dot{\xi}_p - \frac{\lambda_0^2 x_p\xi_p}{2\rho^3}\right) + \lambda_1^2\frac{x_i\epsilon}{r^3} + \tfrac{7}{2}\lambda_1^2\lambda_0^2\frac{\xi_i}{r\rho^3}$$
$$+ \lambda_1^2\frac{x_i}{r^3}\left[\frac{\rho^2}{\rho} + \tfrac{3}{2}\frac{\epsilon^2}{r^2}\right].$$

We can, in the terms of the second order, safely neglect the excentricity and the inclination of the moon's orbit. Then, if

$$D = l_{\mathbb{C}} - l_{\odot}$$

be the difference of the mean longitudes of the moon and sun, we have

$${\sum_p}'x_p\xi_p = r\rho\cos D, \quad \epsilon = r\rho\nu_0\sin D, \quad {\sum_p}'\dot{x}_p\dot{\xi}_p = r\rho\nu\nu_0\cos D,$$

where we can take

$$\nu = \lambda_1 a^{\frac{3}{2}}, \quad \nu_0 = \lambda_0\rho^{\frac{3}{2}}.$$

The terms in the second line of (85) are periodic in D, and can therefore certainly be neglected. The terms in the last line contain secular terms. The secular part of ϵ^2 is

$$\epsilon^2 = \tfrac{1}{2}r^2\rho^2\nu_0^2 = \tfrac{1}{2}\lambda_0^2\frac{r^2}{\rho}.$$

The two terms can therefore be combined, and can be transported to the left-hand member. We then have

$$(86) \quad \frac{d^2x_i}{c^2dt^2} + \lambda_1^2\left(1+\mu-\tfrac{7}{4}\frac{\lambda_0^2}{\rho}\right)\frac{x_i}{r^3} = 4\lambda_1^4\frac{x_i}{r^4} + 4\lambda_1^2\frac{\dot{x}_i\dot{r}}{r^2} - \lambda_1^2\frac{x_i}{r^3}\phi^2.$$

The right-hand member is the same as for an infinitesimal planet. The only secular effect is thus a motion of the perigee amounting to

$$\delta\varpi = 3\frac{\lambda_1^2}{p}v.$$

The motion in one century is

$$0''.06.$$

b. Many of the terms of the second group are zero because the origin of the system of reference has been taken in the sun. We find

$$(87)\quad \frac{d^2x_i}{c^2dt^2}+\lambda_1{}^2m\left(\frac{x_i}{\Delta^3}-\frac{\xi_i}{\rho^3}\right)=4\lambda_1{}^4m\frac{\xi_i}{\rho^4}+4\lambda_1{}^4m^2\left(\frac{x_i}{\Delta^4}-\frac{\xi_i}{\rho^4}\right)$$
$$+4\lambda_1{}^2m\left(\frac{\Delta\dot{x}_i}{\Delta^2}-\frac{\dot{\rho}\dot{\xi}_i}{\rho^2}\right)-\lambda_1{}^2m\left(\frac{x_i}{\Delta^3}\phi_1{}^2-\frac{\xi_i}{\rho^3}\phi_0{}^2\right)-4\lambda_1{}^4m\mu\frac{x_i}{\Delta^4}.$$

The left-hand member contains the ordinary perturbing function. In the usual developments of this function the factor $\lambda_0{}^2/\rho^3$ is replaced by $\nu_0{}^2$. This, however, is not correct in the new theory, but we must take, according to (52, B),

$$\lambda_1{}^2m=\lambda_0{}^2=\rho^3\nu_0{}^2\left(1+\frac{3\lambda_0{}^2}{\rho}\right).$$

The first term on the right contains no co-ordinates of the moon, and can thus only give rise to periodic terms, which we neglect. The last term could be added to the second. It is, however, about 27 million times smaller than this, and can be neglected.

The other terms in the right-hand member produce perturbations in the motion of the moon, which will be investigated in the next article.

c. The terms of the third group are

$$(88)\quad \frac{d^2x_i}{c^2dt^2}=4\lambda_1{}^4m\left(\frac{x_i}{r\Delta^3}-\mu\frac{\xi_i}{r\rho^3}\right)-2\lambda_1{}^4m\left(\frac{x_i}{\rho\Delta^3}-\mu\frac{\xi_i}{\Delta\rho^3}\right)$$
$$+4\lambda_1{}^4m\frac{x_i}{r^3}\left(\frac{1}{\Delta}+\frac{\mu}{\rho}\right)-2\lambda_1{}^4m\frac{x_i}{r^3}\left(\frac{1}{\rho}+\frac{\mu}{\Delta}\right).$$

The last four terms have x_i/r^3 as a factor. If we develop the coefficient in powers of the parallax a/ρ, we can neglect all but the lowest power, and we can thus in these small terms take $\Delta=\rho$. The four terms together then give

$$2\lambda_1{}^2\frac{\lambda_0{}^2}{\rho}\frac{x_i}{r^3}(1+\mu).$$

This can be transported to the left-hand member. The coefficient of x_i/r^3 then becomes

$$\lambda_1{}^2\left(1+\mu-\frac{15}{4}\frac{\lambda_0{}^2}{\rho}-2\mu\frac{\lambda_0{}^2}{\rho}\right),$$

for which we can write

$$\lambda_1{}^2(1+\mu'),$$

and the difference between μ and μ' is very much smaller than the uncertainty of the numerical value of μ. Of the other terms of (88) the third is of the same nature as the second of (87), but 328000 times as small, and therefore entirely negligible. The fourth term is still smaller, and if we replace Δ by ρ, it is of the same nature as the first of (87), and can be neglected for the same reason. The second term, which contains the factor μ and is therefore very small, can, moreover, produce only periodic perturbations. Of all the terms in (88) the first is the largest. It is still about 800 times smaller than the second of (88), and can be neglected. It can, however, also very easily be taken into account. Developing in powers of $1/\rho$, we find for the lowest power

$$-2\lambda_1^2\lambda_0^2\frac{x_i}{r\rho^3} = -2\lambda_1^2\nu_0^2\frac{x_i}{r}.$$

The resulting radial and transversal perturbing forces are (if we neglect the inclination of the moon's orbit),

$$S = -2\nu_0^2, \qquad T = 0.$$

The effect on the motion of the perigee is

$$ed\varpi = 2\nu_0^2 r^2 \cos vdv,$$

which gives only extremely small periodic terms, that can be neglected.

There thus remains only the equation (87), *i.e.* the effect of the pure superposition of the fields of the earth and the sun.

25. We can write this equation in the following form:—

$$\frac{d^2x_i}{c^2dt^2} + \lambda_1^2 m\left(\frac{\mathrm{x}_i}{\Delta^3} - \frac{\xi_i}{\rho^3}\right) = \lambda_1^2[4\mathrm{A}_i + 4\mathrm{B}_i - \mathrm{C}_i], \tag{89}$$

where

$$\mathrm{A}_i = \lambda_1^2 m^2\left(\frac{\mathrm{x}_i}{\Delta^4} - \frac{\xi_i}{\rho^4}\right),$$

$$\mathrm{B}_i = \lambda_1^2 m\left(\frac{\Delta\dot{\Delta}\dot{\mathrm{x}}_i}{\Delta^3} - \frac{\rho\dot{\rho}\dot{\xi}_i}{\rho^3}\right),$$

$$\mathrm{C}_i = \lambda_1^2 m\left(\frac{\mathrm{x}_i}{\Delta^3}\phi_1^2 - \frac{\xi_i}{\rho^3}\phi_0^2\right).$$

We must develop A, B, and C in powers of r/ρ. We will only retain such terms as can give secular perturbations, and of these only those of the lowest degree in r/ρ. We will write x, y, z for x_1, x_2, x_3, and similarly x, y, z, etc.; and for A_1, A_2, A_3 we will write A_x, A_y, A_z, and similarly B_x, . . ., etc. As plane of xy we take the orbital plane of the moon, thus $z = 0$. Then the radial, transversal, and orthogonal perturbing forces are

$$\mathrm{S}_\mathrm{A} = \frac{x}{r}\mathrm{A}_x + \frac{y}{r}\mathrm{A}_y, \quad \mathrm{T}_\mathrm{A} = \frac{x}{r}\mathrm{A}_y - \frac{y}{r}\mathrm{A}_x, \quad \mathrm{W}_\mathrm{A} = \mathrm{A}_z,$$

and similarly for B and C. Let i and Ω be the inclination and ascending node of the moon's orbit on the ecliptic. We will neglect i^2. Also we neglect the excentricity of the earth. Then we have:

$$\xi = \rho \cos L, \qquad \dot{\xi} = -\rho\nu_0 \sin L.$$
$$\eta = \rho \sin L, \qquad \dot{\eta} = \rho\nu_0 \cos L.$$
$$\zeta = -\rho \sin i \sin (L - \Omega), \qquad \dot{\zeta} = -\rho\nu_0 \sin i \cos (L - \Omega).$$

$$\rho^3 {\nu_0}^2 = {\lambda_0}^2 = {\lambda_1}^2 m.$$

Further, we have

$$\Delta^2 = \rho^2 \left[1 + 2\frac{x \cos L + y \sin L}{\rho} + \frac{r^2}{\rho^2} \right],$$

and

$$x = \xi + x, \quad y = \eta + y, \quad z = \zeta.$$

Taking first A_x, we find for the terms of the lowest degree in r/ρ,

$$\frac{x}{\Delta^4} - \frac{\xi}{\rho^4} = \frac{x}{\rho^4} - 4\frac{\xi(x \cos L + y \sin L)}{\rho^5}.$$

The secular part (*i.e.* the part independent of L) of the numerator of the second term is $\frac{1}{2}\rho x$, therefore

$$A_x = -{\lambda_0}^2 m \frac{x}{\rho^4} = -\frac{m{\nu_0}^2}{\rho} x.$$

Similarly we find

$$A_y = -\frac{m{\nu_0}^2}{\rho} y.$$

Further,

$$\frac{z}{\Delta^4} - \frac{\zeta}{\rho^4} = \zeta\left(\frac{1}{\Delta^4} - \frac{1}{\rho^4}\right) = 4\rho \sin i \sin (L - \Omega)\frac{x \cos L + y \sin L}{\rho^5}.$$

The numerator of the last fraction is

$$r \cos (w - L),$$

where w is the moon's longitude. The secular part thus becomes

$$\frac{2r \sin i \sin (w - \Omega)}{\rho^4}.$$

Now we put

$$z' = r \sin i \sin (w - \Omega),$$

so that z' is the co-ordinate of the moon perpendicular to the ecliptic. Then we have

$$A_z = 2\frac{m{\nu_0}^2}{\rho} z'.$$

Consequently

$$(90)\qquad \begin{cases} S_A = -\dfrac{m{v_0}^2}{\rho} r, \\ T_A = 0, \\ W_A = 2\dfrac{m{v_0}^2}{\rho} z'. \end{cases}$$

It is not necessary to relate the computations of the terms B and C at the same length. In B we have $\rho = 0$, so there remains only one term to be developed. We have $\Delta\dot{\Delta} = x\dot{x} + y\dot{y} + z\dot{z}$. After development we find:

$$B_x = \tfrac{1}{2}\frac{m{v_0}^2}{\rho} x - \tfrac{1}{2}\frac{mv_0}{\rho}\dot{y},$$

$$B_y = \tfrac{1}{2}\frac{m{v_0}^2}{\rho} y + \tfrac{1}{2}\frac{mv_0}{\rho}\dot{x},$$

$$B_z = -\tfrac{1}{2}\frac{m{v_0}^2}{\rho} z' + \tfrac{1}{2}\frac{mv_0}{\rho} z'\nu.$$

The factor z' in the last term of B_z is found by neglecting the moon's excentricity in the expressions for $\dot{x}$ and $\dot{y}$, which is allowed on account of the factor $\sin i$. With the same accuracy we can take $\nu = \dot{\theta}$. Then, since

$$x\dot{y} - y\dot{x} = r^2\dot{\theta}, \quad x\dot{x} + y\dot{y} = r\dot{r},$$

we find

$$(91)\qquad \begin{cases} S_B = \tfrac{1}{2}\dfrac{m{v_0}^2}{\rho} r - \tfrac{1}{2}\dfrac{mv_0}{\rho} r\dot{\theta}, \\ T_B = \tfrac{1}{2}\dfrac{mv_0}{\rho}\dot{r}, \\ W_B = -\tfrac{1}{2}\dfrac{m{v_0}^2}{\rho} z' + \tfrac{1}{2}\dfrac{mv_0}{\rho} z'\dot{\theta}. \end{cases}$$

Finally, for C we find similarly

$$C_x = -\tfrac{1}{2}\frac{m{v_0}^2}{\rho} x + \frac{mv_0}{\rho}\dot{y},$$

$$C_y = -\tfrac{1}{2}\frac{m{v_0}^2}{\rho} y - \frac{mv_0}{\rho}\dot{x},$$

$$C_z = \tfrac{3}{2}\frac{m{v_0}^2}{\rho} z' - \frac{mv_0}{\rho} z'\nu,$$

and

$$(92)\qquad \begin{cases} S_C = -\tfrac{1}{2}\dfrac{m{v_0}^2}{\rho} r + \dfrac{mv_0}{\rho} r\dot{\theta}, \\ T_C = -\dfrac{mv_0}{\rho}\dot{r}, \\ W_C = \tfrac{3}{2}\dfrac{m{v_0}^2}{\rho} z' - \dfrac{mv_0}{\rho} z'\dot{\theta}. \end{cases}$$

Collecting results, we find for the total perturbing forces $(4A + 4B - C)$,

$$(93)\qquad \begin{cases} S = -\frac{3}{2}\frac{m{\nu_0}^2}{\rho}r - 3\frac{m\nu_0}{\rho}r\dot{\theta}, \\ T = \qquad\qquad + 3\frac{m\nu_0}{\rho}\dot{r}, \\ W = +\frac{9}{2}\frac{m{\nu_0}^2}{\rho}z' + 3\frac{m\nu_0}{\rho}z'\dot{\theta}. \end{cases}$$

It has already been remarked that the ordinary Newtonian perturbing function, which stands in the left-hand member of (89), also requires a correction. We call the last two terms in this left-hand member

$$-{\lambda_1}^2 R_i,$$

and we develop R_x, R_y, R_z in powers of r/ρ as we have done for A, B, C. Then transporting to the right-hand member, we find for the secular terms of the lowest degree,

$$R_x = \frac{1}{2}\frac{mx}{\rho^3}, \qquad R_y = \frac{1}{2}\frac{my}{\rho^3}, \qquad R_z = -\frac{3}{2}\frac{mz'}{\rho^3}.$$

Therefore

$$S_R = \frac{1}{2}\frac{m}{\rho^3}r, \qquad T_R = 0, \qquad W_R = -\frac{3}{2}\frac{m}{\rho^3}z'.$$

These are the perturbing functions as used in the ordinary lunar theories. Usually m/ρ^3 is replaced by ${n_0}^2$, by Kepler's third law. Now, however, we must take by (52) of the first paper

$$\frac{{\lambda_1}^2 m}{\rho^3} = {\nu_0}^2\left(1 + \frac{3{\lambda_0}^2}{\rho}\right).$$

The correction therefore is

$$\delta\frac{m}{\rho^3} = 3\frac{{\lambda_0}^2}{{\lambda_1}^2}\frac{{\nu_0}^2}{\rho} = 3\frac{m{\nu_0}^2}{\rho}.$$

Therefore

$$(94)\qquad \delta S_R = \frac{3}{2}\frac{m{\nu_0}^2}{\rho}r, \qquad \delta T_R = 0, \qquad \delta W_R = -\frac{9}{2}\frac{m{\nu_0}^2}{\rho^3}z'.$$

It will be seen that these exactly cancel the first terms of (93). There then remain only the second terms, viz.:

$$(95)\qquad S = -3\frac{m\nu_0}{\rho}r\dot{\theta}, \qquad T = 3\frac{m\nu_0}{\rho}\dot{r}, \qquad W = 3\frac{m\nu_0}{\rho}z'\dot{\theta}.$$

26. We found in art. 24 that the new terms in the motion of the moon are completely given by the superposition of the fields of the sun and the earth, both computed as if the other did not exist. The expression for these fields which has been used above is that given by the formulas B of art. 11 (first paper). If we had

used the system A of the same article, we would have found, instead of (89),

$$(96) \qquad \frac{d^2x_i}{c^2dt^2} + \lambda_1{}^2 m\left(\frac{x_i}{\Delta^3} - \frac{\xi_i}{\rho^3}\right) = \lambda_1{}^2[2A_i + 2B_i - 2C_i + 3D_i],$$

where

$$D_i = m\left[\frac{x_i\dot{\Delta}}{\Delta^3} - \frac{\xi_i\dot{\rho}}{\rho^3}\right].$$

It is easily found that the term D is of a higher order in r/ρ than A, B, and C, and can consequently be neglected. The other terms give for S, T, and W exactly the values (95). On the other hand, in this system Kepler's third law is exact (if we neglect the moon's excentricity), as is shown by (52, A), and therefore the terms (94) do not exist in this case. The two systems thus give the same result, as they ought to do.

We must now find the effect of the perturbing forces (95) on the elements of the moon's orbit. We can in the perturbing forces use the Keplerian values for the co-ordinates and velocities. We thus have

$$r\dot{\theta} = \frac{\lambda_1\sqrt{p}}{r}, \qquad \dot{r} = \frac{\lambda_1 e \sin v}{\sqrt{p}}.$$

Substituting these values, we find that secular terms arise in the elements ϖ and Ω. We find

from S: $$ed\varpi = 3\frac{mv_0}{\rho}\lambda_1\sqrt{p}\,.\,r\cos v\,dv,$$

from T: $$ed\varpi = 3\frac{mv_0}{\rho}\frac{\lambda_1}{\sqrt{p}}r^2\left(1+\frac{r}{p}\right)e\sin^2 v\,dv,$$

from W: $$\sin i\,d\Omega = 3\frac{mv_0}{\rho}\frac{\lambda_1}{\sqrt{p}}r^2\sin i\sin^2(w-\Omega)dv,$$

where $w = v + \varpi$. It would not be difficult to integrate these equations rigorously. We get a sufficient approximation, however, by neglecting the excentricity and the inclination. We then find the following secular terms—

$$(97) \qquad \delta\varpi = \delta\Omega = \tfrac{3}{2}\frac{mv_0\nu}{\rho}\,.\,nt,$$

where we have put $a^{\frac{3}{2}}\lambda_1 = \nu$. The numerical value is

$$+1''\cdot 91 \text{ per century,}$$

to which for the perigee we must add the $0''\cdot 06$ found above.

The observed values are *

Brown, Cowell	. .	$\delta\varpi = +14643536'' \pm 2''$
Newcomb, de Vos	. .	$+14643530'' \pm 2''$
Newcomb, Brown	. .	$\delta\Omega = -6967944'' \pm 2''$

* See De Sitter, "The Motions of the Lunar Perigee and Node, and the Figure of the Moon," *Proc. Acad. Amsterdam*, vol xvii. p. 1309 (April 1915).

The theoretical values, including the new terms found here, are

$$\delta\varpi = +14643534'' \pm 2''$$
$$\delta\Omega = -6967939'' \pm 2''$$

It is to be regretted that it is not possible from Brown's lunar theory to find the theoretical motion with one more decimal place. The residuals

$$\Delta\varpi = \begin{cases} +2'' \pm 3'' \\ -4'' \pm 3'' \end{cases}$$
$$\Delta\Omega = \quad -5'' \pm 3''$$

are undoubtedly partly due to the inaccuracy of the theoretical value, owing to accumulation of errors by the neglect of the decimal of the second in the many terms of which the complete value is the sum. The residuals as they stand are therefore not decisive either for or against Einstein's theory.

27. In the equations of art. 21 we will assume that there is only one body, the sun, to which we will assign the index 0, Further, we put

$$\mathrm{x}_i = x_i - x_{i0}, \quad x_{i0} = a_i t, \quad \mathrm{r}^2 = \sum_p{}' \mathrm{x}_p{}^2,$$
$$\sum_p{}' a_p{}^2 = q^2, \quad \sum_p{}' a_p \mathrm{x}_p = -\mathrm{r}\epsilon.$$

Then

$$\frac{\partial^2 \mathrm{r}}{c^2 \partial t^2} = \frac{q^2 - \epsilon^2}{\mathrm{r}}$$

and the g_{ij} become

(98) (D)
$$\begin{cases} -1+\gamma & 0 & 0 & -2a_1\gamma \\ 0 & -1+\gamma & 0 & -2a_2\gamma \\ 0 & 0 & -1+\gamma & -2a_3\gamma \\ -2a_1\gamma & -2a_2\gamma & -2a_3\gamma & 1+\gamma+\frac{1}{2}\gamma^2+\gamma(2q^2-\frac{1}{2}\epsilon^2) \end{cases}$$

where

$$\gamma = -\frac{2\lambda_0{}^2}{\mathrm{r}}.$$

This will be called the system D.

In the first paper we found the field of the sun at rest at the origin of the system of co-ordinates, *i.e.* for $a_i = 0$. The line-element in this gravitational field was

(99) $$ds^2 = -(1-\gamma_0)d\sigma^2 + (1+\gamma_0+\tfrac{1}{2}\gamma_0{}^2)c^2dt^2,$$

where

$$\gamma_0 = -\frac{2\lambda_0{}^2}{r}, \quad r^2 = \sum_p{}' x_p{}^2, \quad x_{i0} = 0.$$

If to (99) we apply a Newton-transformation

$$x_i' = x_i - a_i t,$$

and then after the transformation we drop the accents, we find $\gamma_0 = \gamma$, and the g_{ij} become

$$(100)\quad (\mathrm{N})\quad \begin{cases} -1+\gamma & 0 & 0 & \alpha_1(1-\gamma) \\ 0 & -1+\gamma & 0 & \alpha_2(1-\gamma) \\ 0 & 0 & -1+\gamma & \alpha_3(1-\gamma) \\ \alpha_1(1-\gamma) & \alpha_2(1-\gamma) & \alpha_3(1-\gamma) & 1+\gamma+\frac{1}{2}\gamma^2-q^2(1-\gamma) \end{cases}$$

This will be called the system N. It will be seen that it does not agree with the system D (98). In (N) the g_{i4} are of the order $\frac{1}{2}$, and the difference of g_{44} with its value in (99) is of the first order; in (D) these terms are of the orders $\frac{3}{2}$ and 2 respectively. The velocity of light in the two systems is

$$(101)\quad \begin{cases} (\mathrm{D}) & v = c(1+\gamma), \\ (\mathrm{N}) & v = c(1+\gamma+q\cos \mathrm{V}+\frac{1}{2}q^2\cos^2 \mathrm{V}), \end{cases}$$

where V is the angle between the ray of light and the velocity q.

If, on the other hand, we transform (99) by a Lorentz-transformation,* we find exactly the system D. This is due to the fact that the Lorentz-transformation is an *orthogonal* transformation of the four co-ordinates x_1, x_2, x_3, x_4.

The equations of motion in the systems (D) and (N) are different, also the course of rays of light is different. The *observable* phenomena must be exactly the same in the two systems. It will be interesting to demonstrate this by a simple example.

Let a planet move in a circle, and let it be observed by an observer on the sun. If the radius of the circle be a, then the differential equation for θ is in the two systems

$$(102)\quad \begin{cases} (\mathrm{D}) & \dfrac{d^2\theta}{c^2dt^2} = -\dfrac{\lambda^2\epsilon}{a^2}\dot{\theta}, \\ (\mathrm{N}) & \dfrac{d^2\theta}{c^2dt^2} = 0. \end{cases}$$

* The formulas for a Lorentz-transformation are given in my paper of 1911 on the (old) principle of relativity, *M.N.*, vol. lxxi. p. 391. It must be remarked that what is here called α_1, α_2, α_3 is the same as αq, βq, γq in that paper. By the application of these formulas, and the transformation formula (3) of the first paper (vol. lxxvi. p. 701) the g_{ij} become

$$\mathrm{A}\quad \begin{cases} -1+\gamma_0 & 0 & 0 & -2\alpha_1\gamma_0 \\ 0 & -1+\gamma_0 & 0 & -2\alpha_2\gamma_0 \\ 0 & 0 & -1+\gamma_0 & -2\alpha_3\gamma_0 \\ -2\alpha_1\gamma_0 & -2\alpha_2\gamma_0 & -2\alpha_3\gamma_0 & 1+\gamma_0+\frac{1}{2}\gamma_0{}^2+2q^2\gamma_0. \end{cases}$$

Now, however, γ_0 is not the same as γ, since in γ_0 the co-ordinates x_i and x_{i0} are simultaneous relatively to the time t, and in γ relatively to the time t'. We must then apply the formula (6) of vol. lxxi. p. 393, where the index 1 must be taken to refer to the planet and 2 to the sun. Then $-\alpha q_1\Delta_2 = \frac{1}{2}\alpha_1 r\epsilon$ in our present notation, and, since $\sum_p{}'\alpha^p x^p = -r\epsilon$, we have, accurate to the second degree in q, $\mathrm{r}^2 = r^2(1-\epsilon^2)$, and consequently $\gamma_0 = \gamma(1-\frac{1}{2}\epsilon^2)$. This reduces (A) to (98).

The light which leaves the planet at the time t reaches the observer at a time t_0, which is different in the two systems, since the velocity of light is different. In both systems, however, we can apply the rule of planetary aberration, which asserts that the true direction at the time t is the apparent direction at the time t_0. In both systems, therefore, the observed value θ_0 is the same as the true value θ belonging to the time t, as derived from the equation (102). We can thus omit the index o of θ_0.

The observer does not know the time t, but only the time t_0. We must therefore write the differential equation with respect to this time. We thus find

$$\frac{d^2\theta}{c^2 dt_0^2} = \frac{d^2\theta}{c^2 dt^2}\left(\frac{dt}{dt_0}\right)^2 + \frac{d\theta}{c^2 dt}\frac{d^2t}{dt_0^2}.$$

In the system (D) the velocity of light is the same in all directions, consequently $t_0 - t$ is a constant, and

$$(103) \qquad (D)\quad \frac{d^2\theta}{c^2 dt_0^2} = \frac{d^2\theta}{c^2 dt^2} = -\frac{\lambda^2\epsilon}{a^2}\dot{\theta}.$$

In the system (N) we have, to the first order in q,

$$v = c(1 + q\cos V) = c(1 + \epsilon),$$

and consequently

$$t_0 = t + \int_0^a \frac{dr}{v} = t + \frac{a}{c}(1 - \epsilon).$$

Therefore

$$\frac{d^2t}{dt_0^2} = \frac{a}{c}\frac{d^2\epsilon}{dt_0^2} = ac\frac{d^2\epsilon}{c^2 dt^2} = -\frac{c\lambda^2\epsilon}{a^2}.$$

Consequently in the system (N) we find

$$(103') \qquad (N)\quad \frac{d^2\theta}{c^2 dt_0^2} = -\frac{\lambda^2\epsilon}{a^2}\dot{\theta},$$

or exactly the same as in the system (D).

28. In this article we will consider the gravitational field produced by the fixed stars. We will suppose that the distribution of the stars in space is symmetrical round the sun. Of course this is very far from being true in the actual galactic system, but, if we took a distribution in ellipsoidal, instead of spherical, shells we would find practically the same result, and also for more irregular distributions the results would undoubtedly be of the same order of magnitude. Since exact numerical results are, of course, entirely out of the question, and the order of magnitude is all we can hope to derive, it is not necessary at present to consider any but the simplest possible case.

We will thus compute the field produced by a hollow spherical shell, whose total mass may be M, and whose inner and outer radii are R_1 and R_2 respectively. The analysis of art. 11 of the

first paper can be used unaltered, and—as it will appear that we can neglect terms having λ^4 as a factor—we will find

$$ds^2 = -d\sigma^2 + c^2dt^2 + \gamma(d\sigma^2 + c^2dt^2),$$

where

$$\gamma = -\frac{2\lambda^2}{r}m(r) + 2\lambda^2[n(r) - \mathrm{N}].$$

Integrating by parts we find

$$m(r) = rn(r) - s(r),$$

where

$$s(r) = \int_0^r n(r)dr.$$

Therefore

$$\gamma = -2\lambda^2\mathrm{N} + \frac{2\lambda^2}{r}s(r). \tag{104}$$

Inside the shell we have $s(r) = 0$, and outside $s(r) = r\mathrm{N} - \mathrm{M}$. If the density ρ were constant we would have $s(r) = \frac{1}{2}m(r)$. If ρ decreases with increasing r, the ratio $s(r)/m(r)$ will increase. We thus have always

$$m(r) \leqq 2s(r), \tag{105}$$

and $m(r)$ and $s(r)$ will remain of the same order of magnitude so long as ρ differs appreciably from zero. It will be remembered that $m(r)$ is the total mass within the sphere of radius r.

Inside the shell we have

$$\gamma_0 = -2\lambda^2\mathrm{N}.$$

Therefore γ_0 is a constant, and there will be *no* gravitational effects. For, if we perform the transformation

$$x_i' = x_i\sqrt{(1-\gamma_0)}, \quad t' = t\sqrt{(1+\gamma_0)},$$

the g_{ij} will be reduced to the values given by

$$ds^2 = -d\sigma^2 + c^2dt^2.$$

The units of space and time will be changed, but this cannot be verified by observations of the motions of bodies inside the shell relatively to each other.

The difference in the unit of time can, however, be verified by observing light which emanates from a point within the mass, or outside the shell. The frequency of light-vibrations is proportional to $\sqrt{(1+\gamma)}$ [see first paper, art. 13]. Now $s(r)$ is always positive, therefore all light coming from distances exceeding R_1 must appear displaced *towards the violet* as compared with light from a source inside the shell. If γ_1 be the value of γ at the source of the light, then the ratio of the frequencies is

$$\frac{\mathrm{N}_1}{\mathrm{N}_0} = \sqrt{\frac{1+\gamma_1}{1+\gamma_0}}. \tag{106}$$

Now there is certainly no systematic displacement of the stellar spectral lines towards the violet. Of the very faint stars, of which no measures of spectral lines can be made, we know, of course, nothing; but of stars within a certain distance, say r_1, we can say with certainty that a systematic displacement towards the violet is smaller than, say, $\frac{1}{3}$ km./sec. On the displacement towards the violet produced by the general field is, of course, superposed the displacement towards the red, produced by each star separately (*cf.* art. 13). This may easily cancel the displacement towards the violet. It is well known that stars of all spectra actually show a small systematic displacement towards the red. Still, to be quite safe, we will increase the limit, and only assert that the general field of the stars certainly does not cause a displacement exceeding 1 km./sec. towards the violet. We can then say

$$\sqrt{\frac{1+\gamma_1}{1+\gamma_0}} < 1 + 3 \cdot 10^{-6}$$

or

$$\gamma_1 - \gamma_0 < 6 \cdot 10^{-6}(1+\gamma_0). \tag{107}$$

In (106) no assumption is made as to the smallness of γ_1 or γ_0; (107) shows, however, that they must be small, and consequently the neglect of the terms of the second order is justified. Now we have

$$\gamma_1 - \gamma_0 = \frac{2\lambda^2}{r} s(r).$$

Therefore, dividing by $\lambda^2 = 10^{-8}$, and using (105), we have

$$m(r_1) < 600\, r_1.$$

Here r_1 is expressed in astronomical units. If we express it in parsecs we have, roughly,

$$m(r_1) < 10^8 \cdot r_1. \tag{108}$$

If we take, *e.g.*, for r_1 the distance corresponding to a parallax of $0''{\cdot}01$ we would find

$$m(0''{\cdot}01) < 10^{10}.$$

It appears probable that the actual mass within this distance is much smaller. It should be pointed out that, although the limit has been derived from the observations of stellar spectra, $m(r)$ includes *all* mass within a sphere of radius r. It thus also includes all faint or dark stars, and all nebulæ and meteoric or other matter.

29. Having now discussed the principal astronomical consequences of the new theory, we will return briefly to the fundamental concepts. The principle of general relativity asserts that the space co-ordinates and the time are entirely irrelevant, and have no physical meaning whatever. In fact, we never observe anything but a combination of time and space. Nobody has ever measured a

pure distance, nor a pure interval of time. Always the distance, or the interval of time, is measured from a material point (event) m_1 at time t_1 and at the place (x_1, y_1, z_1) to another point m_2 at another time t_2 and another place (x_2, y_2, z_2).

Let us, by way of illustration, consider the concept "velocity." The differential quotient $\frac{d\sigma}{dt}$ does not correspond to any physical reality. It is a consequence of the theory, or of the old theory of relativity, that no material velocity can exceed the velocity of light. This does not mean that $\frac{d\sigma}{dt}$ can never exceed the value c, but that its value for a material body at any point of the four-dimensional time-space and in any direction is always smaller than the value of the same differential quotient for a ray of light at the same point and in the same direction. In other words, no material body can ever overtake a ray of light, and every moving body can be overtaken by light.

Take, *e.g.*, two systems of co-ordinates A and B. The system A we will suppose to have been so chosen relatively to existing material bodies that at large distances from these bodies the geodetic lines are practically straight lines. Such a system is by Einstein called a *Galilean* system. Let the system B be non-Galilean. Then B must be derived from A, and A from B, by a non-linear transformation. In both systems we take cylindrical co-ordinates r, θ, z, and let the transformation which reduces the one system to the other be a rotation. Thus, if θ' refers to B and θ to A, we have

$$\theta' = \theta - \omega ct.$$

The values of the line-element in the two systems, if we neglect the gravitational effect of material bodies, are

$$(109) \qquad \begin{cases} \text{A. } ds^2 = -d\sigma^2 + c^2dt^2, \\ \text{B. } ds^2 = -d\sigma'^2 - 2r^2\omega d\theta' cdt + (1 - r^2\omega^2)c^2dt^2, \end{cases}$$

where $d\sigma^2 = dr^2 + r^2d\theta^2 + dz^2$, and similarly $d\sigma'$.

Now consider a "velocity" perpendicular to the radius-vector in the plane $z = 0$, thus $d\sigma = \pm rd\theta$. The velocity of light in the two systems is

$$\text{A. } v = c,$$
$$\text{B. } v' = c(1 \mp r\omega),$$

where the upper sign must be taken for light travelling in the direction of increasing θ'. It follows that for values of r exceeding $1/\omega$ light is propagated tangentially in only one direction. Similarly we can show that for $r\omega > 1$ it is a physical impossibility for a material body to be at rest in the system B. It must necessarily move towards the negative θ' with a velocity exceeding $(r\omega - 1)c$, and smaller than $(r\omega + 1)c$. This shows the irreality of the co-ordinates and of the "velocity." Still, it may very well be that,

for smaller values of r, the system B is more appropriate for the description of the actual phenomena than A.

The difference between the two systems is characterised by the value of g_{24}.* In A we have $g_{24}=0$ and in B $g_{24}=-r^2\omega$. Both values are solutions of the equation

$$G_{24}=0,$$

which, of course, is the same in all systems of co-ordinates. The right-hand member is zero, since we neglect the presence of matter. It can be shown that the equation is satisfied by

$$g_{24}=kr^2,$$

k being an arbitrary constant. The theory only gives the differential equation, and does not prescribe the constant of integration. This must, of course, be so determined that the solution represents the observed relative motions of material bodies and light-rays as described in the adopted system of co-ordinates. The particular solution which does so in one system must, if this system is transformed into another, by the same transformation be reduced to the particular solution which fits the observed phenomena in the new system. This implies that the constants of integration are also subjected to the transformation, and are therefore as a rule different in different systems of co-ordinates.†

In classical mechanics the constants of integration are prescribed, since only one system of co-ordinates is admitted. This prescription is usually given in the form of the values of the variable quantities at infinity. Thus the solution of Poisson's equation is subjected to the condition that the potential shall be zero at infinity.‡ In the theory of general relativity the values at infinity are, of course, different in different systems of co-ordinates.

If we consider physical phenomena at the surface of the earth, and describe them in a system of co-ordinates in which this surface

* We need only consider g_{24}. We could, *e.g.*, introduce the condition (12) $g=-1$, or any other, then g_{44} would follow immediately from g_{24}.

† We must be careful to distinguish between the indeterminateness of the constants of integration (or rather the arbitrary functions, or boundary conditions) corresponding to the fact that the g_{ij} are given by *differential* equations, and the fundamental indeterminateness involved by the principle of general relativity, which has been explained in art. 8 (first paper), and which allows a free choice of the system of co-ordinates, and of the measure of space and time. Thus, *e.g.*, in art. 10 the assumption $g_{i4}=0$ is of mixed character. The primary assumption was that we should have a system possessing three-dimensional spherical symmetry, and in which the field should be stationary. This imposes certain conditions on the g_{i4}, as well as on the other g_{ij}, which might, if desired, be expressed in the form of differential equations. That among all possible solutions of these equations we chose $g_{i4}=0$ is of the nature of a boundary condition.

‡ Of course we know nothing about infinity, and never will. The true condition determining the arbitrary constants or functions is that the actually observed phenomena shall be correctly represented, and only for mathematical convenience this condition is brought in the form of a boundary condition for infinity.

is fixed, we find that the formula (109, B) is the one we need to represent the observations [apart from gravitational effects], and not (109, A), if only we take for ω the appropriate value. This value we call the rotational velocity of the earth.

Suppose we use a system of co-ordinates relatively to which the earth has a velocity of rotation ω_1. This means that the co-ordinate θ of any point on the earth would in that system be represented by the formula $\theta = \theta_0 + \omega_1 t$. This ω_1 is, of course, entirely arbitrary. It is also not ascertainable by observations, since the co-ordinate axes cannot be observed. It must thus in a true theory of relativity disappear from the final equations describing the observed or observable phenomena. The differential equation gives $g_{24} = kr^2$, and the constant k is also undetermined, and must also disappear from the final equations. Now, if we come to compare the theory with the observations, we will find that, in order to make it fit, we have to take

$$k = \omega_1 - \omega,$$

and if this is introduced, both k and ω_1 will disappear and only the observable quantity ω will remain.*

Rotation in its purely kinematical aspect is thus relative in the new theory, quite as relative as linear translation. We can transform it away as we can transform away linear translation. But there is a difference. If a linear translation is transformed away (by a Lorentz-transformation), it is utterly gone; no trace of it remains. Not so in the case of rotation. The transformation which does away with a rotation at the same time alters the equation of relative motion in a definite manner. This shows that rotation is not a purely kinematical fact, but an essential physical reality. Its amount ω is a physical constant, proper to the earth, like its mass.

I have purposely in the above only referred to such methods of ascertaining and measuring ω which depend exclusively on observations made on the surface of the earth (Foucault's pendulum, gyroscopes, the ellipsoidal figure of the earth), without any reference to celestial bodies. That by all these methods, and also by the apparent daily motion of the stars, we find the same value of ω, is a confirmation of the correctness of the theories by means of which the various phenomena, which are used in the different methods, are explained, especially of the theory of inertia. But all this is entirely independent of the system of co-ordinates.

The case is somewhat analogous with the gravitational effect on planetary motion of a rotation of the sun. Whether the solar system has a translation or not can only be ascertained by observations of bodies outside that system. Therefore such a translation can have no effect on the relative motions within the system.

* See also De Sitter, "The Relativity of Rotation in Einstein's Theory," *Acad. of Sci., Amsterdam*, Sept. 1916 (vol. xix. p. 527).

But whether the sun rotates or not can be ascertained by observations within the solar system, and accordingly the principle of relativity does admit an effect of the rotation of the sun on the relative motions of the planets.

30. Newton in his *Principia* gave only formal laws for gravitation and inertia. He made no attempt at explanation: "Hypotheses non fingo." When Newton's integral law of gravitation was replaced by Poisson's equation, the concept of a *field* of gravitation, of which the material bodies were the sources, was introduced. At the same time the absolute space was substantialised and called "ether." At very large distances from any material body the potential is zero: the whole of the numerical value of the potential at any point can thus be derived from the material sources.

Expressed in the language of Einstein's theory, we would say, at infinity the g_{ij} are:

$$(110) \qquad \begin{Vmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & +1 \end{Vmatrix}$$

These are the natural values, or the g_{ij} of absolute space, or of the ether. In classical mechanics the g_{ij} in the neighbourhood of matter are the same as (110), with the exception of g_{44}, which becomes

$$g_{44} = 1 + \gamma,$$

and γ is produced by the material bodies through Poisson's equation. In Einstein's theory all g_{ij} differ from the values (110), and they are all determined by differential equations, of which the right-hand members (κT_{ij}) depend on matter. Thus matter here also appears as the source of the g_{ij}, *i.e.* of inertia. But can we say that the *whole* of the g_{ij} is derived from these sources? The differential equations determine the g_{ij} apart from constants of integration, or rather arbitrary functions, or boundary conditions, which can be mathematically defined by stating the values of g_{ij} at infinity. Evidently we could only say that the whole of the g_{ij} is of material origin if these values at infinity were *the same for all systems of co-ordinates.* It is not necessary to insist on the values (110). We can prescribe any other set of degenerated values which the g_{ij} must have at infinity, only they must be the same for all systems of reference, *i.e.* they must be invariant for all transformations, or at least for a so comprehensive group of transformations that restriction to this group does not mean giving up the principle of relativity.

The values (110) are certainly not invariant. Einstein has, however,* pointed out a set of degenerated g_{ij} which are actually

* In a conversation with the writer.

invariant for all transformations in which, at infinity, x_4 is a pure function of x_4'. They are:

$$(111) \qquad \left\{ \begin{matrix} 0 & 0 & 0 & \infty \\ 0 & 0 & 0 & \infty \\ 0 & 0 & 0 & \infty \\ \infty & \infty & \infty & \infty^2 \end{matrix} \right.$$

Einstein makes the hypothesis that at infinity the g_{ij} actually have the values (111). These are then the "natural" values, and any deviation from them must be due to material sources. The meaning of this degenerated set of g_{ij} is that at infinity the four-dimensional time-space is dissolved into a three-dimensional space and a one-dimensional time. In the three-dimensional space all transformations are still allowed, also those involving arbitrary functions of t: rotations, accelerated motions, etc. At infinity we would thus have an absolute time,* but no absolute space. Of course, so far as our observations reach (in space and time), there would still be complete four-dimensional relativity, but at very large distances from all matter the g_{ij} would gradually converge towards the degenerated values (111).

The hypothesis can thus be said to be consistent with the formal principle of relativity. It is, however, not a consequence of it, and also denying it is not contrary to the principle. If we adopt the hypothesis, then in any system of co-ordinates the g_{ij} would have exactly determined values, there being no constants of integration left by which they can be made to fit the observed phenomena. Now it is certain that, in many systems of reference (*e.g.* in all Galilean systems), the g_{ij} at large distances from all material bodies known to us actually have the values (110). On Einstein's hypothesis these are special values which, since they differ from (111), must be produced by some material bodies. Consequently there must exist, at still larger distances, certain unknown masses which are the source of the values (110), *i.e.* of all inertia. These hypothetical masses would form the boundary of the universe, which would thus be necessarily finite and limited. Outside them there would be *nothing* but the field of g_{ij} gradually degenerating to the values (111).

We must insist on the impossibility that any of the known fixed stars or nebulæ can form part of these hypothetical masses. The light even from the farthest stars and nebulæ has approximately the same wave-length as light produced by terrestrial sources.

* This must be so understood that we are free to introduce any other time t', which is an arbitrary function of t, but not of x_1, x_2, x_3. For finite values of x_1, x_2, x_3, x_4 *all* transformations are still allowed, but they must all at infinity degenerate in such a way that t' becomes independent of the space-co-ordinates. Thus, *e.g.*, an ordinary Lorentz-transformation would be no longer allowed, since it does not degenerate at infinity; but a transformation which for all finite values of x_1, x_2, x_3, x_4 coincides with a Lorentz-transformation, while degenerating at infinity in the required manner, would be allowable.

Consequently the ratio of the units of time and space is in these nebulæ approximately the same as here; the deviation of the g_{ij} from the Galilean values (110) is of the same order as here, and they must therefore be still inside the limiting envelope which separates our universe from the outer parts of space, where the g_{ij} have the values (111).

If the hypothesis is accepted, then rotation is relative not only in its kinematical aspect, but also physically. The constant ω in the value of g_{24} is then induced by, or derived from, these distant masses, and it is at the same time the velocity of rotation relatively to these masses. These thus take up the rôle of the absolute space in the old theory of inertia. And, to my mind at least, they are quite as objectionable as this absolute space, if not more so. If we believe in the existence of these supernatural masses, which control the whole physical universe without having ever been observed, then the temptation must be very great indeed to give preference to a system of co-ordinates relatively to which they are at rest, and to distinguish it by a special name, such as "inertial system" or "ether." * Formally the principle of relativity would remain true, but as a matter of fact we would have returned to the absolute space under another name.

Theoretically it is, of course, important that the *possibility* † has been shown to ascribe the whole of the deviation of the actual g_{ij} from certain standard values to material sources, but evidently this "explanation" of the origin of inertia by hypothetical masses which must be imagined for that purpose and have not been otherwise observed, is practically equivalent to no explanation at all, or to a confession of our ignorance. The hypothesis is not an essential element of Einstein's theory. We can accept that theory without the hypothesis. We need not, therefore, give up the hope that some day a satisfactory explanation of inertia will be found which, of course, must, as all physical theories, be consistent with the principle of relativity. In the meantime, we can be content to put ourselves again on the point of view of Newton, accepting the formal law and making no hypotheses.

Even if Einstein has not explained the origin of inertia and gravitation, his theory represents an enormous progress over the physics of yesterday. Perceiving the irrelevance of the representation by co-ordinates in which our science was clothed, he has penetrated to the deeper realities which lay hidden behind it, and not only has he entirely explained the exceptional and universal nature of gravitation by the principle of the identity of gravitation and inertia, but he has laid bare intimate connections between

* Of course, once the source of inertia had been pointed out, the need would be felt to explain *how* the inertia is transferred from these distant masses to the known material bodies, and a hypothetical mechanism would probably be built up quite analogous to the ether in the old theories.

† At least the *logical* possibility. Whether it will be possible to imagine an actual distribution of ordinary physical matter having the desired effect, remains to be investigated. If at all finite, they must certainly enormously exceed the whole known physical universe.

branches of science which up to now were considered as entirely independent from each other, and has thus made an important step towards the unity of nature. Finally, his theory not only explains all that the old theory of relativity could explain (experiment of Michelson, etc.), but, *without introducing any new hypothesis or empirical constant*, it explains the anomalous motion of the perihelion of Mercury, and it predicts a number of phenomena which have not yet been observed. It has thus at once proved to be a very powerful instrument of discovery.

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